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A FAMILY OF PERIODIC ORBITS IN THE LIÉNARD DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we investigate the existence of a family of periodic orbits for the Liénard systems by expressing these systems as a Hamiltonian system with two degrees of freedom The main technique applied here is the averaging theory.

1. Introduction and statement of the main results

The Liénard oscillator is defined by the equation.

$$\ddot{x} + \phi(x)\dot{x} + \psi(x) = \varphi(t), \tag{1}$$

and is frequently used as a nonlinear model in various scientific fields, depending on the specific forms of $\phi(x)$, $\psi(x)$, and $\varphi(t)$. For example, equation (1) reduces to the Van der Pol oscillator, a well-known nonlinear model for electronic oscillations, when $\phi(x) = \mu(x^2 - 1), \psi(x) = x$, and $\varphi(t) = 0$, see [4][10]. Additionally, nonlinear evolution equations, such as the Burgers-KdV equation, which emerge from many physical phenomena, can also be transformed into equation (1) see [3]. Consequently, the study of equation (1) is of great physical and theoretical importance.

In this work, we will analytically investigate the existence of periodic orbits in Liénard differential systems, which take the following form

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$$\begin{cases} \ddot{x} - \mu\left(1 + \sum_{k=1}^{n} a_k x^k\right)\dot{x} + x = 0, \\ \ddot{y} + \mu\left(1 + \sum_{k=1}^{n} a_k x^k\right)\dot{y} + y = 0. \end{cases} \tag{2}$$

Where μ is sufficiently small and $a_k \neq 0$.

Let $p_y = \dot{x}$ and $p_x = \dot{y} + \mu(1 + \sum_{k=1}^{n} a_k x^k)y$, then, the second-order ordinary nonlinear differential equations (2) can be expressed as a Hamiltonian system with following form

$$\begin{aligned} \dot{x} &= p_y \\ \dot{y} &= p_x - \mu\left(1 + \sum_{k=1}^{n} a_k x^k\right)y \\ \dot{p}_x &= -y + \mu\left(\sum_{k=1}^{n} k a_k x^{k-1}\right)y p_y \\ \dot{p}_y &= -x + \mu\left(1 + \sum_{k=1}^{n} a_k x^k\right)p_y \end{aligned} \tag{3}$$

the Hamiltonian corresponding to the system (3) can be obtained as

$$\mathcal{H}\left(x, y, p_x, p_y\right) = p_x p_y + xy - \mu\left(1 + \sum_{k=1}^{n} a_k x^k\right)y p_y. \tag{4}$$

Recall that p_x and p_y are the conjugate momenta.

Numerous papers in the literature focused on the analytical studies of periodic solutions for Hamiltonian systems using averaging theory; see references [7][1]. In the paper [5], a novel analytical study to studying the periodic solutions of a Hamiltonian system using averaging theory was presented, using a different methodology.

In this work, we aim to study the periodic orbits of a Hamiltonian system (3) resulting from second-order ordinary nonlinear differential equations (2), analytically using averaging theory. Periodic orbits of Hamiltonian systems or differential systems are usually investigated numerically due to the difficulty of analytical study, as the latter can sometimes be impossible. With the help of fresh concepts and methodologies, the methodology used in averaging theory can be expanded to investigate the periodic orbits of many other Hamiltonian systems using new ideas and methods. Our main results are the following.

Theorem 1. *For μ sufficiently small, $a_k \neq 0$ with $k \in \mathbb{N}^*$ and for each $h \in \mathbb{R}$ the Liénard differential systems (2) have at most $\left[\frac{n}{2}\right]$ periodic orbits.*

Here $[\cdot]$ denotes the integer part function.

2. The averaging theory of first order

In this section, we summarize the key results from the first-order averaging theory to compute periodic orbits, which will be used to prove Theorem 1. The proofs for the following results can be found in Theorems 11.5 and 11.6 of Verhulst's book [9], with further improvements in [2] and [6]. Further information on the general theory of averaging may be found in the monograph [8].

We consider the differential equation

$$\dot{x} = \mu \mathrm{F}(t, x) + \mu^2 \mathrm{G}(t, x, \mu), \text{ with } x(0) = x_0, \tag{5}$$

where μ is a small parameter, $x \in \Omega \subset \mathbb{R}^n$ is an open subset and $t \in \mathbb{R}^+$. We assume that the functions $F(x,t)$ and $G(t,x,\eta)$ are periodic in t of period T.

Now, we consider the averaged differential equation associated with Eq. (5)

$$\dot{y} = \mu f(y), \text{ with } y(0) = x_0, \tag{6}$$

where $y \in \Omega$. As usual, we denote the first-order averaged function by $f(y)$, defined as

$$f(y) = \frac{1}{T}\int_0^T \mathrm{F}(t,y)dt \tag{7}$$

Theorem 2. *For the Eq. (5) and (6), we make the following assumptions:*

(i) F is a C^2 function and is bounded by a constant independent of μ in $[0,\infty) \times U$ for all $\mu \in (0,\mu_0]$.

(ii) F and G are T-periodic in t, where T is independent of μ.

(iii) The solution $y(t)$ of Eq. (6) belongs to U for all t in the interval $\left[0,\frac{1}{\mu}\right]$

If p is an equilibrium point of the Eq. (6), where

$$\det\left(\frac{\partial f}{\partial y}\right)\Big|_{y=p} \neq 0.$$

there exists a T-periodic solution $x(t,\mu)$ of Eq. (5) such that $x(t,\mu) \to p$ as $\mu \to 0$.

3. Proof of Theorem 1

It is evident that the Hamiltonian system's (3) only equilibrium point is the origin (0,0,0,0). The characteristic polynomial $p(\lambda)$ for the linearized of Hamiltonian system (3) at the origin is

$$p(\lambda) = \lambda^4 + (2-\mu^2)\lambda^2 + 1,$$

and its four eigenvalues are given by

$$\pm\frac{\mu}{2} \pm \frac{\sqrt{\mu^2-4}}{2}.$$

By direct computing, we obtain the matrix of the linear part of the system (3) at the origin (0,0,0,0) as follows

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\mu & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}$$

and its real Jordan normal form is

$$\begin{pmatrix} \frac{\mu}{2} & -\frac{\sqrt{4-\mu^2}}{2} & 0 & 0 \\ \frac{\sqrt{4-\mu^2}}{2} & \frac{\mu}{2} & 0 & 0 \\ 0 & 0 & -\frac{\mu}{2} & -\frac{\sqrt{4-\mu^2}}{2} \\ 0 & 0 & \frac{\sqrt{4-\mu^2}}{2} & -\frac{\mu}{2} \end{pmatrix}$$

With respect to Hamiltonian system (3), we perform a linear transformation

$$\begin{aligned} x &= -\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y, \\ p_x &= U, \\ y &= -\frac{\sqrt{4-\mu^2}}{2}Z + \frac{\mu}{2}U, \\ p_y &= Y, \end{aligned}$$

where μ is a small parameter. By expressing the Hamiltonian system (3) in new variables (X, Y, Z, U), we get the following new system

$$\begin{aligned} \dot{X} &= \frac{1}{2\sqrt{4-\mu^2}}\Big(-(4-\mu^2)Y + \mu\sqrt{4-\mu^2}X \\ &\quad +2\mu^2 \textstyle\sum_{k=1}^{n} a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^k Y\Big), \\ \dot{Y} &= \frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y \\ &\quad +\mu \textstyle\sum_{k=1}^{n} a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^k Y, \\ \dot{Z} &= -\frac{\sqrt{4-\mu^2}}{2}U - \frac{\mu}{2}Z \\ &\quad -\mu \textstyle\sum_{k=1}^{n} a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^k Z \\ &\quad + \frac{\sum_{k=1}^{n} a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^k \mu^2 U}{\sqrt{4-\mu^2}} \\ &\quad -\frac{\mu^2}{2}YZ \textstyle\sum_{k=1}^{n} k a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^{k-1} \\ &\quad +\frac{\mu^3}{2\sqrt{4-\mu^2}}YU \textstyle\sum_{k=1}^{n} k a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^{k-1}, \\ \dot{U} &= \frac{\sqrt{4-\mu^2}}{2}Z - \frac{\mu}{2}U \\ &\quad -\frac{\mu}{2} \textstyle\sum_{k=1}^{n} k a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^{k-1} Y\sqrt{4-\mu^2}Z \\ &\quad +\frac{\mu^2}{2} \textstyle\sum_{k=1}^{n} k a_k \left(-\frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y\right)^{k-1} YU. \end{aligned} \tag{8}$$

The Hamiltonian (4) can be rewritten in terms of the new variables (X, Y, Z, U) as follows

$$
\begin{aligned}
H = {} & YU + \left(-\tfrac{\sqrt{4-\mu^2}}{2}X + \tfrac{\mu}{2}Y\right)\left(-\tfrac{\sqrt{4-\mu^2}}{2}Z + \tfrac{\mu}{2}U\right) \\
& -\mu\left(1 + \textstyle\sum_{k=1}^{n} a_k \left(-\tfrac{\sqrt{4-\mu^2}}{2}X + \tfrac{\mu}{2}Y\right)^k\right) \\
& \times \left(-\tfrac{\sqrt{4-\mu^2}}{2}Z + \tfrac{\mu}{2}U\right)Y,
\end{aligned} \tag{9}
$$

In the differential system (8) we change the variables (X, Y, Z, U) to (r, θ, s, ϕ), where

$$
\begin{aligned}
X &= r\cos\theta \\
Y &= r\sin\theta \\
Z &= s\cos(\theta+\phi), \\
U &= s\sin(\theta+\phi),
\end{aligned}
$$

with $r > 0$ and $s > 0$. In the new variables, system (8) becomes

$$
\begin{aligned}
\dot{r} = {} & \tfrac{r}{4}\big(2 - 4\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k(\cos(\theta))^2 \\
& +4\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k\big)\mu + O(\mu^2) \\
\dot{\theta} = {} & 1 + \cos(\theta)\sin(\theta)\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k\mu + O(\mu^2) \\
\dot{s} = {} & \tfrac{s}{2\cos(\theta)}(-\cos(\theta) + 2\sin(\theta+\phi) \\
& \times\big(\textstyle\sum_{k=1}^{n} a_k k(-r\cos(\theta))^k\sin(\theta)\cos(\theta+\phi)\big) \\
& -2\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k(\cos(\theta+\phi))^2\cos(\theta)\big)\mu + O(\mu^2) \\
\dot{\phi} = {} & \tfrac{1}{\cos(\theta)}\big(-(\cos(\theta))^2\sin(\theta)\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k \\
& +\sin(\theta+\phi)\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k\cos(\theta+\phi)\cos(\theta) \\
& +(\cos(\theta+\phi))^2\textstyle\sum_{k=1}^{n} a_k k(-r\cos(\theta))^k\sin(\theta)\big)\mu + O(\mu^2)
\end{aligned} \tag{10}
$$

In the new variables (r, θ, s, ϕ) the first integral H given in (9) rewritten as follows

$$
\begin{aligned}
H = {} & r\sin(\theta)s\sin(\theta+\phi) + \left(-1/2\sqrt{4-\mu^2}r\cos(\theta) + 1/2\mu\, r\sin(\theta)\right) \\
& \times\left(-1/2\sqrt{4-\mu^2}s\cos(\theta+\phi) + 1/2\mu\, s\sin(\theta+\phi)\right) \\
& -\mu\left(1 + \textstyle\sum_{k=1}^{n} a_k\left(-1/2\sqrt{4-\mu^2}r\cos(\theta) + 1/2\mu\, r\sin(\theta)\right)^k\right) \\
& \times\left(-1/2\sqrt{4-\mu^2}s\cos(\theta+\phi) + 1/2\mu\, s\sin(\theta+\phi)\right)r\sin(\theta)
\end{aligned} \tag{11}
$$

Next, by taking θ as the new variable for time, the differential system (10) can be expressed in the following form.

$$\begin{aligned}
&\frac{dr}{d\theta} = r' = \frac{r}{4}\big(2 - 4\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k(\cos(\theta))^2 \\
&+4\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k\big)\mu + \mathcal{O}(\mu^2) \\
&\frac{ds}{d\theta} = s' = \frac{s}{2\cos(\theta)}(-\cos(\theta) + 2\sin(\theta+\phi) \\
&\times \big(\textstyle\sum_{k=1}^{n} a_k k(-r\cos(\theta))^k \sin(\theta)\cos(\theta+\phi)\big) \\
&-2\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k(\cos(\theta+\phi))^2\cos(\theta)\big)\mu + \mathcal{O}(\mu^2) \\
&\frac{d\phi}{d\theta} = \phi' = \frac{1}{\cos(\theta)}\big(-(\cos(\theta))^2\sin(\theta)\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k \\
&+\sin(\theta+\phi)\textstyle\sum_{k=1}^{n} a_k(-r\cos(\theta))^k\cos(\theta+\phi)\cos(\theta) \\
&+(\cos(\theta+\phi))^2\textstyle\sum_{k=1}^{n} a_k k(-r\cos(\theta))^k\sin(\theta)\big)\mu + \mathcal{O}(\mu^2)
\end{aligned} \tag{12}$$

From the first integral H in the new variables (r, θ, s, ϕ) (11) and $H = h$, we get

$$\phi = \arccos\left(\frac{h}{rs}\right) \tag{13}$$

Substituting ϕ into differential system (12) gives that

$$\begin{aligned}
r' &= \frac{r}{4}\big(2 - 4\textstyle\sum_{k=1}^{n} (-1)^k a_k r^k(\cos(\theta))^{k+2} \\
&\quad +4\textstyle\sum_{k=1}^{n} (-1)^k a_k r^k(\cos(\theta))^k\big)\mu + \mathcal{O}(\mu^2), \\
&= \mu \mathrm{F}_{11}(\theta, r, s) + \mathcal{O}(\mu^2), \\
s' &= \frac{s}{2\cos(\theta)}\left(-\cos(\theta) + 2\sin\left(\theta + \arccos\left(\frac{h}{rs}\right)\right)\right. \\
&\quad \times\left(\textstyle\sum_{k=1}^{n} (-1)^k a_k k r^k(\cos(\theta))^k\sin(\theta)\cos\left(\theta + \arccos\left(\frac{h}{rs}\right)\right)\right) \\
&\quad \left.-2\textstyle\sum_{k=1}^{n} (-1)^k a_k r^k(\cos(\theta))^k\left(\cos\left(\theta + \arccos\left(\frac{h}{rs}\right)\right)\right)^2\cos(\theta)\right)\mu \\
&\quad +\mathcal{O}(\mu^2), \\
&= \mu \mathrm{F}_{12}(\theta, r, s) + \mathcal{O}(\mu^2).
\end{aligned} \tag{14}$$

The differential system (14) has been written into the standard form (5) to apply the averaging theory. Here using the notation of Theorem 2, the following should be considered

$$\begin{aligned}
&x = y = (r, s),\ t = \theta,\ T = 2\pi \\
&\mathrm{F}(t, x) = \binom{\mathrm{F}_{11}(\theta, r, s)}{\mathrm{F}_{12}(\theta, r, s)}, f(x) = \binom{f_{11}(r, s)}{f_{12}(r, s)},
\end{aligned}$$

and we have

$$\begin{aligned}
f_{11}(r, s) &= \frac{r}{4}\left(2 - \frac{2}{\pi}\textstyle\sum_{k=1}^{n} (-1)^k a_k r^k \int_0^{2\pi} (\cos(\theta))^{k+2} d\theta\right. \\
&\quad \left.+\frac{2}{\pi}\textstyle\sum_{k=1}^{n} (-1)^k a_k r^k \int_0^{2\pi} (\cos(\theta))^k d\theta\right) \\
&= \frac{r}{4}\left(2 - \frac{2}{\pi}\textstyle\sum_{k=2}^{n} (-1)^k a_k r^k(\alpha_{k+2} - \alpha_k)\right) \\
&= p(r)
\end{aligned}$$

and

$$f_{12}(r,s) = \frac{1}{2\pi}\int_0^{2\pi} \frac{s}{2\cos(\theta)}\left(-\cos(\theta) + \sin\left(2\theta + 2\arccos\left(\frac{h}{rs}\right)\right)\right.$$
$$\times\left(\sum_{k=1}^{n} (-1)^k a_k k r^k (\cos(\theta))^k \sin(\theta)\right)$$
$$\left.-2\sum_{k=1}^{n} (-1)^k a_k r^k (\cos(\theta))^k \left(\cos\left(\theta + \arccos\left(\frac{h}{rs}\right)\right)\right)^2 \cos(\theta)\right) d\theta,$$

After simplifying the function $f_{12}(r,s)$, it becomes as follows

$$\begin{aligned}
f_{12}(r,s) = -\frac{s}{4\pi}\Bigg(1 &- 2\sum_{\substack{k=2\\ \text{k even}}}^{n} (2k+1)(-1)^k a_k r^k \left(\frac{h}{rs}\right)^2 \int_0^{2\pi} (\cos(\theta))^k d\theta \\
&+4\sum_{\substack{k=2\\ \text{k even}}}^{n} (k+1)(-1)^k a_k r^k \left(\frac{h}{rs}\right)^2 \int_0^{2\pi} (\cos(\theta))^{k+2} d\theta \\
&+2\sum_{\substack{k=2\\ \text{k even}}}^{n} (-1)^k a_k r^k \int_0^{2\pi} (\cos(\theta))^k (k+1) d\theta \\
&-2\sum_{\substack{k=2\\ \text{k even}}}^{n} (k+1)(-1)^k a_k r^k \int_0^{2\pi} (\cos(\theta))^{k+2} d\theta\Bigg), \\
= -\frac{s}{4\pi}\Bigg(1 &- 2\sum_{\substack{k=2\\ \text{k even}}}^{n} (2k+1)(-1)^k a_k r^k \left(\frac{h}{rs}\right)^2 \alpha_k \\
&+4\sum_{\substack{k=2\\ \text{k even}}}^{n} (k+1)(-1)^k a_k r^k \left(\frac{h}{rs}\right)^2 \alpha_{k+2} \\
&+2\sum_{\substack{k=2\\ \text{k even}}}^{n} (k+1)(-1)^k a_k r^k \alpha_k \\
&-2\sum_{\substack{k=2\\ \text{k even}}}^{n} (k+1)(-1)^k a_k r^k \alpha_{k+2}\Bigg),
\end{aligned}$$

Now, we rely on the following integration

$$\alpha_k = \begin{cases} \int_0^{2\pi} (\cos(\theta))^k d\theta \neq 0 & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases}$$

It is easy to verify that the polynomial $p(r)$ has at most $\left[\frac{n}{2}\right]$ positive roots. For these roots, the function $f_{12}(r,s)$ accepts at most one positive root for the variable s. Moreover, if the determinant of the Jacobian matrix (7) for our differential system (14) is non-zero at these roots, then by Theorem 2, for μ sufficiently small, and for each $h \in \mathbb{R}$, there can be at most $\left[\frac{n}{2}\right]$ periodic orbits. Thus, we can conclude that for μ sufficiently small, if $a_k \neq 0$ for $k \in \mathbb{N}^*$ and for each $h \in \mathbb{R}$ the Liénard differential systems (2) have at most $\left[\frac{n}{2}\right]$ periodic orbits.

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