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HIGHER ORDER OSCILLATION AND UNIFORM DISTRIBUTION

Shigeki Akiyama[1] — Yunping Jiang[2,3]

[1]University of Tsukuba, Ibaraki, JAPAN

[2]Queens College of CUNY Flushing, NY, USA

[3]The CUNY Graduate School, New York, NY, USA

ABSTRACT. It is known that the Möbius function in number theory is higher order oscillating. In this paper we show that there is another kind of higher order oscillating sequences in the form $(e^{2\pi i\alpha\beta^n g(\beta)})_{n\in\mathbb{N}}$, for a non-decreasing twice differentiable function g with a mild condition. This follows the result we prove in this paper that for a fixed non-zero real number α and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers α) and for all real polynomials $Q(x)$, sequences $\big(\alpha\beta^n g(\beta) + Q(n)\big)_{n\in\mathbb{N}}$ are uniformly distributed modulo 1.

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1. Introduction

We denote by $\mathbb{N}$ the set of positive integers. Suppose $\mathbf{c} = (c_n)_{n\in\mathbb{N}}$, is a sequence of complex numbers. In [2] (see also [4]), an oscillating sequence is defined for the purpose of the study of Sarnak's conjecture (see [8, 9]) which is stated as the Möbius function is linearly disjoint from all zero entropy flows. Let us recall the definition of an oscillating sequence.

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Definition 1 (Oscillation). The sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ is said to be oscillating if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n e^{2\pi int}=0,\ \ \forall\, 0\le t<1. \tag{1}$$

with the technical condition

$$\sum_{n=1}^{N}|c_n|^{\lambda}=O(N)\quad\text{ for some }\quad\lambda>1. \tag{2}$$

Recall that the Möbius function $\mu(n)$ is, by definition, $\mu(n)=1$ if $n=1$; $\mu(n)=(-1)^r$ if $n=p_1\cdots p_r$ for r distinct prime numbers p_i; $\mu(n)=0$ if $p^2|n$ for some prime number p. The Möbius sequence $\mathbf{u}=(\mu(n))_{n\in\mathbb{N}}$ is the one generated by the Möbius function. Due to Davenport's theorem [1], the Möbius sequence is oscillating.

We proved in [2] that any oscillating sequence is linearly disjoint from all minimally mean attractable (MMA) and minimally mean-L-stable (MMLS) flows. In the same paper, we further proved that flows defined by all p-adic polynomials of integral coefficients, all p-adic rational maps with good reduction, all automorphisms of the 2-torus with zero topological entropy, all diagonalizable affine maps of the 2-torus with zero topological entropy, all orientation-preserving circle homeomorphisms are MMA and MMLS. Furthermore, in [4], we proved that flows defined by all continuous interval maps with zero topological entropy are MMA and MMLS. Therefore, we confirmed Sarnak's conjecture for these flows which form a large class of zero topological entropy flows. However, it is also shown in [2, Example 7], only the oscillation property is not enough for the study of Sarnak's conjecture. We need a higher order oscillation condition in the study of Sarnak's conjecture. We gave a definition of a higher order oscillating sequence in [5].

Definition 2 (Higher order oscillation). We call the sequence $\mathbf{c}=(c_n)_{n\in\mathbb{N}}$ a higher order oscillating sequence of order $m\ge 2$ if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n e^{2\pi iP(n)}=0 \tag{3}$$

for every real polynomial P of degree $\le m$ with also the technical condition (2).

We prove in [5] that any higher order oscillating sequence of order d is linearly disjoint from all affine distal flows on the d-torus for all $d\ge 2$. One consequence of this result is that any higher order oscillating sequence of order 2 is linearly disjoint from all affine flows on the 2-torus with zero topological entropy. Thanks to Hua's result [3], the Möbius sequence $\mathbf{u}$ is an example of a higher order

oscillating sequence of order m for all $m \geq 2$ (see also [7, Lemma 2.1] and [10]). Combining this with our main result in [5], we reconfirmed in [5] that Sarnak's conjecture for all affine flows on the 2-torus with zero topological entropy and for all affine distal flows on the d-torus for all $d > 2$ by using a much simple method. Then we have the following interesting question.

Question 1. Is there another kind of higher order oscillation sequences as defined in Definition 2 except for the one generated by an arithmetic function like the Möbius function?

We study this question in this paper.

2. Statement of the main result

For a real number x, let $[x]$ denote the integer part of x, that is, the greatest integer $\leq x$; let

$$\{x\} = x - [x]$$

be the fractional part of x, or the residue of x modulo 1.

Definition 3 (Uniform distribution)**.** We say a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed modulo 1 (abbreviated u. d. mod 1) if for any $0 \leq a < b \leq 1$, we have

$$\lim_{N\to\infty} \frac{\#(\{n \in [1, N] \mid \{x_n\} \in [a, b]\})}{N} = b - a.$$

We state three results in the u. d. mod 1 theory.

Theorem A (The Weyl criterion)**.** *The sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is u. d.* mod 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \textit{for all integers } h \neq 0.$$

Theorem B (Koksma's theorem)**.** *Let $\big(y_n(x)\big)_{n\in\mathbb{N}}$ be a sequence of real valued C^1 functions defined on an interval $[a, b]$. Suppose $y'_m(x) - y'_n(x)$ is monotone on $[a, b]$ for any two integers $m \neq n$ and suppose*

$$\inf_{m\neq n} \min_{x\in[a,b]} \big|y'_m(x) - y'_n(x)\big| > 0.$$

Then for almost all $x \in [a, b]$, the sequence $\mathbf{y} = \big(y_n(x)\big)_{n\in\mathbb{N}}$ is u. d. mod 1*.*

Theorem C (Van der Corput's theorem). *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers. If for every positive integer* h *the sequence*

$$\mathbf{d_h x} = (x_{n+h} - x_n)_{n\in\mathbb{N}} \quad \textit{is u. d.} \mod 1,$$

then $\mathbf{x}$ *itself is u. d.* mod 1.

The reader who is interested in these three theorems can find proofs in [6, Theorem 2.1, Theorem 3.1, and Theorem 4.3].

For the convenience of notation, we understand that an empty sum is 0 and an empty product is one, i.e.,

$$\sum_{j=1}^{k}(\cdots) = 0 \quad \text{and} \qquad \prod_{j=1}^{k}(\cdots) = 1 \quad \text{when} \quad k = 0.$$

In this notation, we have

$$y_n(x) = x^n = x^n \prod_{j=1}^{k}(\cdots) \quad \text{and} \quad y_n(x)\sum_{j=1}^{k}(\cdots) = 0 \quad \text{when} \quad k = 0.$$

Take a non empty interval I in the real line $\mathbb{R}$, which can be closed, open or semi-open. Let $\mathcal{C}^k_+(I)$ be the space of all positive real valued k-times continuously differentiable functions on an interval I, whose i-th derivative is non-negative for $i \le k$. Then it is closed under addition and multiplication, that is, if

$$f, g \in \mathcal{C}^k_+(I),$$

then

$$f + g, \quad fg \in \mathcal{C}^k_+(I).$$

In what follows, we often use this *closure property* of $\mathcal{C}^k_+(I)$. Let $\mathbb{R}[x]$ denote the space of all real polynomials. The main result we prove in this paper is

Theorem 1 (Main theorem). *Let us take a function* $g \in \mathcal{C}^2_+((1,\infty))$. *Then, for a fixed real number* $\alpha \neq 0$ *and almost all real numbers* $\beta > 1$ *(alternatively, for a fixed real number* $\beta > 1$ *and almost all real numbers* α*) and for all real polynomials* $Q \in \mathbb{R}[x]$, *sequences*

$$\Big(\alpha\beta^n g(\beta) + Q(n)\Big)_{n\in\mathbb{N}} \quad \textit{are u. d.} \mod 1. \tag{4}$$

As a countable union of exceptional null sets is null, we clearly have

Corollary 1. *Given a countable family* $\{g_i \mid i \in \mathbb{N}\}$ *in* $\mathcal{C}^2_+((1,\infty))$. *Then, for a fixed real number* $\alpha \neq 0$ *and almost all real numbers* $\beta > 1$ *(alternatively, for a fixed real number* $\beta > 1$ *and almost all real numbers* α*) and for all real polynomials* $Q \in \mathbb{R}[x]$, *sequences*

$$\Big(\alpha\beta^n g_i(\beta) + Q(n)\Big)_{n\in\mathbb{N}}, \quad i \in \mathbb{N}, \quad \textit{are u. d.} \bmod 1.$$

Since $x^n - 1 \in \mathcal{C}^2_+((1,\infty))$ for all $n \geq 1$, we have that for any $g \in \mathcal{C}^2_+((1,\infty))$ and any integer $l \geq 0$ and $h_j \in \mathbb{N}$, $1 \leq j \leq l$,

$$g(x)\prod_{j=1}^{l}\big(x^{h_j} - 1\big) \in \mathcal{C}^2_+((1,\infty)).$$

Applying this countable family to Corollary 1, we obtain:

Theorem 2 (Equivalent statement). *Given any* $g \in \mathcal{C}^2_+((1,\infty))$. *Then, for a fixed real number* $\alpha \neq 0$ *and almost all real numbers* $\beta > 1$ *(alternatively, for a fixed real number* $\beta > 1$ *and almost all real numbers* α*), for all integers* $l \geq 0$ *and all* l*-tuple* $(h_1, \ldots, h_l) \in \mathbb{N}^l$, *and for all real polynomials* $Q \in \mathbb{R}[x]$, *sequences*

$$\left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}\big(\beta^{h_j} - 1\big) + Q(n)\right)_{n\in\mathbb{N}} \quad \textit{are u. d.} \bmod 1. \tag{5}$$

On the other hand, when $l = 0$, the product is empty, Theorem 2 is reduced to Theorem 1. So our Theorem 1 and Theorem 2 are equivalent. Our proof is based on the formulation of Theorem 2, which is already an interesting point in this paper. We will give the proof in § 3.

Theorem 1 combined with Theorem A answers the question (Question 1) affirmatively.

Corollary 2 (Main corollary). *Given any* $g \in \mathcal{C}^2_+((1,\infty))$. *Then, for a fixed real number* $\alpha \neq 0$ *and almost all real numbers* $\beta > 1$ *(alternatively, for a fixed real numbers* $\beta > 1$ *and almost all real numbers* α*), sequences*

$$\mathbf{c} = \Big(e^{2\pi i\alpha\beta^n g(\beta)}\Big) \tag{6}$$

are higher order oscillating sequences of order m *for all* $m \geq 2$.

Remark 1. In particular, taking a constant function $g \equiv 1$, we have

$$\mathbf{c} = \Big(e^{2\pi i\alpha\beta^n}\Big)_{n\in\mathbb{N}}.$$

3. Proof of the main theorem

We start the proof for the case that $\alpha \neq 0$ is fixed and figure out the exceptional set for $\beta > 1$. We may assume that $\alpha > 0$ and fixed it from now on. Take $g \in \mathcal{C}^2_+((1,\infty))$. For $n \in \mathbb{N}$ and an integer $l \geq 0$ and for a l-tuple $(h_1, \ldots, h_l) \in \mathbb{N}^l$, define a function

$$y_n(x) = \alpha g(x) x^n \prod_{j=1}^{l} \big(x^{h_j} - 1\big).$$

Remember that when $l = 0$, $y_n(x) = \alpha x^n g(x)$.

For $n > m$, we have

$$\begin{aligned} y_n'(x) - y_m'(x) = \quad & \alpha g(x) \Bigg(\big(nx^{n-1} - mx^{m-1}\big) \prod_{j=1}^{k} \big(x^{h_j} - 1\big) + \\ & \big(x^n - x^m\big) \sum_{j=1}^{k} h_j x^{h_j - 1} \prod_{i \neq j} \big(x^{h_i} - 1\big) \Bigg) + \\ & \alpha g'(x) \big(x^n - x^m\big) \prod_{j=1}^{l} \big(x^{h_j} - 1\big). \end{aligned}$$

Since $g', g'' \geq 0$ and since

$$nx^{n-1} - mx^{m-1} = x^{m-1}\big(nx^{n-m} - m\big)$$

and

$$nx^{n-m} - m \geq n - m \geq 1 \quad \text{for } x \geq 1,$$

we see that every term in the last expression are in $\mathcal{C}^1_+([1,\eta])$ for any $\eta > 1$. By the closure property of $\mathcal{C}^1_+([1,\eta])$, we have that

$$y_n'(x) - y_m'(x) \in \mathcal{C}^1_+([1,\eta]) \quad \forall n > m \in \mathbb{N}. \tag{7}$$

In particular, this imply that $y_n' - y_m'$ is increasing for $n > m \in \mathbb{N}$. Furthermore, if $x > a > 1$, then $x^h - 1 \geq a - 1$ for any $h \in \mathbb{N}$, we see that there is a constant $L > 0$ such that

$$|y_n'(x) - y_m'(x)| \geq L \qquad \forall n > m \in \mathbb{N}, \quad \forall\, a \leq x \leq \eta. \tag{8}$$

Inequalities (7) and (8) say that the sequence of real valued C^1 functions

$$\big(y_n(x)\big)_{n \in \mathbb{N}}$$

satisfies all hypothesizes of Theorem B.

Theorem B implies that for almost all x in

$$[(2^k+1)/2^k, (2^{k-1}+1)/2^{k-1}] \quad \text{or} \quad [k, k+1] \quad \text{for} \ \ k \geq 2,$$

the sequence

$$\big(\alpha y_n(x)\big)_{n\in\mathbb{N}} \quad \text{is u. d. mod } 1.$$

Further, this implies that for almost all

$$x \in (1,\infty) = \cup_{k=2}^{\infty} \left[\frac{2^k+1}{2^k}, \frac{2^{k-1}+1}{2^{k-1}}\right] \cup \cup_{k=2}^{\infty}[k, k+1]$$

the sequence $\big(\alpha y_n(x)\big)_{n\in\mathbb{N}}$ is u. d. mod 1.

Let

$$A_{(h_1,h_2,\ldots,h_l)} = \left\{\beta > 1 \;\middle|\; \left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}\big(\beta^{h_j}-1\big)\right)_{n\in\mathbb{N}} \text{is not u. d. mod } 1\right\}.$$

Then it has one dimensional Lebesgue measure zero. By the above convention, we include the case $l = 0$ as well.

Since the set

$$U = \bigcup_{l=0}^{\infty}\{(h_1,\ldots,h_l) \mid h_j \in \mathbb{N}\}$$

is countable, the one dimensional Lebesgue measure of

$$\bigcup_{(h_1,\ldots,h_l)\in U} A_{(h_1,\ldots,h_l)}$$

is zero, too.

For the fix a real number $\alpha \neq 0$ in the theorem, take a real number

$$\beta \in (1,\infty)\Big\backslash \bigcup_{(h_1,\ldots,h_l)\in U} A_{(h_1,\ldots,h_l)}.$$

This says that the sequence

$$\left(\alpha\beta^n g(\beta)\prod_{j=1}^{l}(\beta^{h_j}-1)\right)_{n\in\mathbb{N}} \quad \text{is u. d. mod } 1$$

for all integers $l \geq 0$ and all l-tuple $(h_1,\ldots,h_l) \in \mathbb{N}^l$.

Define statements $P(k)$ for $k = 0, 1, \dots$ as follows.

$P(k)$: *For any integer* $l \geq 0$,

$$(h_1, \dots, h_l) \in \mathbb{N}^l \qquad \text{and} \qquad t_i \in \mathbb{R}\ (i = 0, 1, 2, \dots, k),$$

the sequence

$$\left(\alpha\beta^n g(\beta) \prod_{j=1}^{l} \left(\beta^{h_j} - 1\right) + \sum_{i=0}^{k} t_i n^i \right)_{n\in\mathbb{N}} \qquad \textit{is u. d} \bmod 1.$$

We claim that $P(k)$ holds for every integer $k \geq 0$. We prove the claim by induction.

By our choice of α and β, we know that $P(0)$ holds.

Assume $P(k-1)$ holds for $k \geq 1$. Let

$$x_n = \alpha\beta^n g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1\right) + \sum_{i=0}^{k} t_i n^i.$$

Then

$$x_{n+h} = \alpha\beta^{n+h} g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1\right) + \sum_{i=0}^{k} t_i (n+h)^i.$$

Consider the difference appeared in Theorem C:

$$\begin{aligned} x_{n+h} - x_n &= \alpha\beta^n(\beta^h - 1) g(\beta) \prod_{j=1}^{\ell} \left(\beta^{h_j} - 1\right) + \sum_{i=0}^{k-1} T_i n^i \\ &= \alpha\beta^n g(\beta) \prod_{j=1}^{\ell+1} \left(\beta^{h_j} - 1\right) + \sum_{i=0}^{k-1} T_i n^i \end{aligned}$$

$$\text{with} \quad h_{\ell+1} = h \qquad \text{and} \qquad T_i = -t_i + \sum_{j=i}^{k} t_j \binom{j}{i} h^{j-i}.$$

Since $P(k-1)$ is valid, the resulting sequence is u. d. mod 1 for all $h \in \mathbb{N}$. Now Theorem C implies that $P(k)$ holds too. We proved the claim. Therefore we completed the proof of Theorem 2.

The proof for the case that $\beta > 1$ is fixed and to obtain the exceptional set for α, is similar and easier. Let g be any positive function on $(1, \infty)$. For $n \in \mathbb{N}$ and an integer $l \geq 0$ and for a l-tuple $(h_1, \dots, h_l) \in \mathbb{N}^l$, define

$$y_n(x) = xg(\beta)\beta^n \prod_{j=1}^{l} \left(\beta^{h_j} - 1\right). \tag{9}$$

Then
$$y'_n(x) - y'_m(x) = g(\beta)\big(\beta^n - \beta^m\big)\prod_{j=1}^{l}\big(\beta^{h_j} - 1\big)$$
is a positive constant for $n > m$ and satisfies the condition of Theorem B under the similar dissection of the interval $(1, \infty)$. The rest of the proof is the same.

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Shigeki Akiyama
Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai, Tsukuba
Ibaraki, 305-8571
JAPAN
E-mail: akiyama@math.tsukuba.ac.jp

Yunping Jiang
Department of Mathematics
Queens College of CUNY
Flushing, NY 11367-1597
USA
Department of Mathematics
The CUNY Graduate School
365 Fifth Avenue
New York, NY 10016
USA
E-mail: yunping.jiang@qc.cuny.edu