uniform distribution theory

DOI: 10.2478/UDT-2025-0001
Unif. Distrib. Theory **20** (2025), no.1, 1-15

VALUATION OF AMERICAN LOOKBACK-OPTIONS AND THE DISTRIBUTION OF KRONECKER SEQUENCES

GERHARD LARCHER[1] AND STEFAN POMBERGER[1]

[1]Institute of Financial Mathematics and Applied Number Theory Johannes Kepler University Linz. Austria

ABSTRACT. In this paper, we determine the complexity of calculating the fair value of an American lookback-option in a binomial n−step model via backwardation. It will turn out, that the complexity also depends on diophantine properties of a certain parameter of the model. Therefore, we will need some basic tools from the theory of uniform distribution for the determination of the complexity.

Communicated by Friedrich Pillichshammer

1. Introduction

It is the goal of this paper to explicitly determine the complexity of finding the fair value of an American lookback-option in a binomial stock-model. This topic and this goal should not scare a potential reader who is not so familiar with or not so interested in mathematical finance! Indeed, this goal reduces to a simple counting problem, and the final result shows some connection to irregularities of distributions, more concrete, to discrepancy estimates for Kronecker sequences.

So a reader with no interest in finance can skip the subsequent Chapter 2 in which we will describe the valuation problem and in which we will reduce it to the counting problem. In Chapter 3 we formulate the problem once more, we give the result (Theorem 1) in the so-called "rational case", and we prove Theorem 1. In Chapter 4 we give the tools from uniform distribution theory which we Will need to handle the "irrational case" and to prove the corresponding Theorem 2 in Chapter 5.

2020 Mathematics Subject Classification: 11K38, 91G20.
Keywords: lookback options, derivative pricing, discrepancy, Koksma-Hlawka-inequality.

2. The valuation of American lookback-options in a binomial stock-model

A *binomial n-step model* is a simple discrete model for the expansion of a stock price. This model gets its importance from the fact, that a geometric Brownian motion (which is a very common continuous stock-price model) can be approximated arbitrarily fine by such binomial models. The parameters which determine a binomial n-step model are (remark for insiders: for simplicity in the following, we restrict to interest rates $r = 0$):

- the number n of steps at times $k \cdot dt$ for $k = 0,1,2,\dots,n-1$
- the starting value S_0, i.e., the price of the stock at time 0 (a positive real number, for simplicity we set $S_0 = 1$ in all the following)
- the constant factors u (for an up-step) and d (for a down-step) (positive real numbers with $d < 1 < u$ (for insiders: this must hold since we assumed $r = 0$, otherwise the model would not be arbitrage-free)
- the constant probability p for an up-step in any of the time points $k \cdot dt$.

With these parameters within this model a stock price S can develop in the following schematic form from time 0 until time $T := n \cdot dt$.

Here each step up appears with probability p and each step down with probability $1 - p$.

An *American lookback-option* D based on this stock now is a financial product with the following property (we present here a special version of such an option, there also exist other versions):

If we possess the product D, then in every time-point $k \cdot dt$ for $k = 0,1,2,\dots,n$ we consider

- the actual price S_k of the stock S, and
- the maximum price $S_k(max)$ attained by the stock between times 0 and $k \cdot dt$,

and we can decide then in every time-point $k \cdot dt$ if we want to receive a payout of quantity $S_k(max) - S_k$ right now (then the product terminates) or to wait to the next time-point.

A basic question in the theory of derivative pricing is: “What is the fair value of this product D at time 0?” This question can be answered with the help of an algorithm, which is called *backwardation*. We will not explain and define this algorithm, an interested reader can find a detailed description and discussion of the algorithm with examples in [6], Volume 2, Chapters 2.10 and 2.11] (see also [2]). For further information concerning valuation of path-depending American options in general and American lookback options in special see [1], [3] or [7].

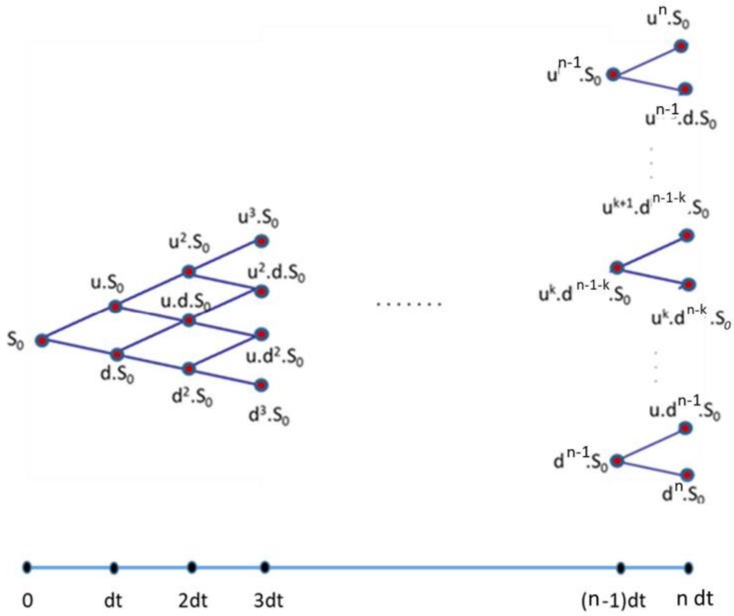

FIGURE 1. Graph of a binomal *n-step* model.

As is pointed out in [6], Volume 2, Chapter 2.10, the complexity of the backwardation-algorithm for an American lookback-option essentially depends on a certain quantity Q which will be defined in the following:

For an arbitrary price-point, say uadbS0 in the binomial model at time $k \cdot dt$ with $k = a + b$(for $0 \le a, b \le n$ with $a + b = k \le n$) let

$$M(a,b) := \#\{u^x d^y S_0 : 0 \le x \le a, 0 \le y \le b, u^x d^y S_0 \ge \max(S_0, u^a d^b S_0)\},$$

i.e., $M(a,b)$ is the number of possible maximum-values on possible paths from S_0 to $u^a d^b S_0$. Then the complexity Q of the algorithm is essentially given by

$$Q = \sum_{\substack{a,b=0\\ a+b\le n}}^{n} M(a,b).$$

A trivial upper estimate for Q obviously is

$$Q \le \sum_{a=0}^{n} \sum_{b=0}^{n-a} (a+1)(b+1) = \frac{(n+1)(n+2)(n+3)(n+4)}{24} = \frac{n^4}{24} + \frac{5n^3}{12} + O(n^2)$$

In the following we want to give the (at least for the leading terms) exact value of Q.

It will turn out, that the value of Q depends on diophantine properties of the following quantity $\alpha := -\frac{\log u}{\log d} = \frac{\log u}{\log e}$, if we set $d =: \frac{1}{e}$. Note, that by the assumption $0 < d < 1 < u$ we have $\alpha > 0$.

The first obvious dependence on Diophantine properties is the following: Note, that $M(a,b)$ is not defined as the *number of knots* in the graph, where a possible maximum is attained, but as the *number of different possible maximum values*!

Hence: If $\alpha = \frac{\log u}{\log e}$ is rational, say $\alpha = \frac{A}{B}$ with relative prime integers A and B, then $u^B d^A = 1$, hence $u^{lB} d^{lA} = 1$ for all integers l and therefore $u^{lB+x} d^{lA+y} = u^x d^y$ for all integers l. That means, that many of the expressions $u^x d^y S_0$ in the definition of $M(a,b)$ have the same value. If α is irrational, then $u^x d^y S_0 = u^{x'} d^{y'} S_0$ for $(x,y) \neq (x',y')$ never can happen. So it seems to be obvious, that the complexity of the algorithm is smaller in the case of a rational α than in the case of an irrational α.

As a second dependence it will turn out, that in the case of an irrational α, also the degree of approximability of α by rationals will have an (slight) influence on the complexity. (Note, that rationality or irrationality of the factor α does not affect the speed of convergence of the binomial model to the geometric Brownian motion.)

In the following we will handle the case of rational α first, and the case of irrational α later.

By the way: We can choose, without loss of generality here and in all the following $S_0 = 1$.

3. The complexity of the algorithm for rational α

Before we start the investigation of the quantity Q we state the problem once more, without any reference to mathematical finance, just as an enumeration problem in a simple graph:

Problem: In a graph as illustrated in Figure 1, determine the following quantity

$$Q = \sum_{\substack{a,b=0 \\ a+b \leq n}}^{n} M(a,b),$$

where:

$$M(a,b) = \#\{u^x d^y S_0 : 0 \leq x \leq a, 0 \leq y \leq b, u^x d^y S_0 \geq \max(S_0, u^a d^b S_0)\}.$$

In this chapter we consider the case $\alpha := -\frac{\log u}{\log d} = \frac{\log u}{\log e} = \frac{A}{B}$ with relative prime integers A and B.

Example 1: As a warm-up – and as a support for the intuition – we first handle the most basic case (which also appears frequently in the concrete application in finance), namely the case, where $d = \frac{1}{u}$, i.e., $\alpha = A = B = 1$. In this case $u^x d^y = u^{x-y}$, hence $u^x d^y = u^{x'} d^{y'}$ iff $x - y = x' - y'$, and the graph of the binomial model then has the following (schematic) form (see Figure 2):

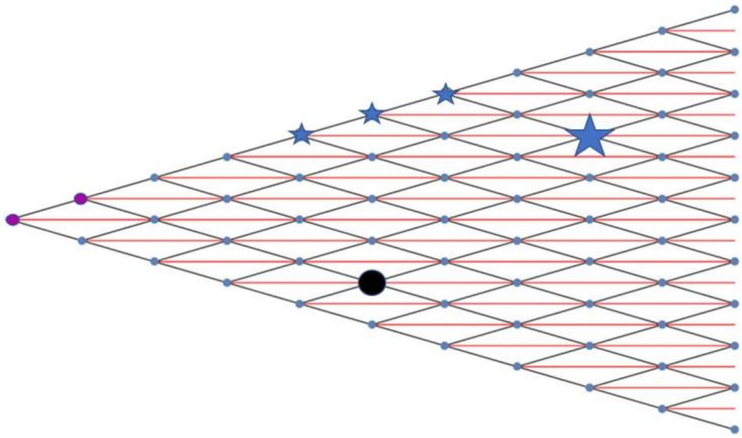

FIGURE 2. Graph of a binomal *n-step* model

In Figure 2 the horizontal lines indicate the different values that can be attained by $u^x d^y$.

For a "big-star" point $u^a d^b$ which is larger or equal to 1, the "small-star" points in the graphic indicate the possible different values $u^x d^y$ with $0 \leq x \leq a,\ 0 \leq y \leq b$, and $u^x d^y \geq u^a d^b$.

For a "big-disk" point $u^a d^b$ which is less than 1, the "small-disc" points (most left two knots on the upper barrier of the model) in the graphic indicate the possible different values $u^x d^y$ with $0 \leq x \leq a,\ 0 \leq y \leq b$, and $u^x d^y \geq 1$.

From Figure 2 it is easy to understand that for a, b such that $u^a d^b$ is larger or equal to 1 (e.g., "big-star" point) the value of $M(a,b)$ (e.g., number of "small-star" points) is given by $b+1$. Also from Figure 2 it is easy to understand that for a, b such that $u^a d^b$ is less than 1 (e.g., "big-disk" point) the value of $M(a,b)$ (e.g., number of "small-disk" points) is given by $a+1$.

Hence for the complexity Q of the algorithm we have

$$Q = \sum_{\substack{a,b=0\\a+b\leq n}}^{n} M(a,b) = \sum_{a=0}^{n} \sum_{\substack{b=0\\b\leq a}}^{n-a} (b+1) + \sum_{a=0}^{n} \sum_{\substack{b=0\\a<b}}^{n-a} (a+1)$$

$$= \sum_{a=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{b=0}^{a} (b+1) + \sum_{a=\left\lfloor\frac{n}{2}\right\rfloor+1}^{n} \sum_{b=0}^{n-a} (b+1) + \sum_{a=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \sum_{b=a+1}^{n-a} (a+1)$$

$$= \begin{cases} \frac{n^3}{12} + \frac{5n^2}{8} + \frac{17n}{12} + 1 & \text{if } n \text{ is even,} \\ \frac{n^3}{12} + \frac{5n^2}{8} + \frac{17n}{12} + \frac{7}{8} & \text{if } n \text{ is odd.} \end{cases}$$

In any case $Q = \frac{n^3}{12} + O(n^2)$.

Now we move to the general case $\alpha = \frac{A}{B}$ with integers A and B with $gcd(A,B) = 1$. We start with a Lemma:

LEMMA 1. *If* $u^a d^b \geq 1$ *then*

$$M(a,b) = \sum_{x=0}^{B-1} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq \frac{u^a}{e^b}}}^{b} 1 + \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq \frac{u^a}{e^b}}}^{A-1} 1 + \mathcal{O}(1) =: \overset{\circledast}{\sum} + \mathcal{O}(1)$$

where the $\mathcal{O}$-constant may depend on A and B only.

a) If $u^a d^b < 1$ then

$$M(a,b) = \sum_{x=0}^{B-1} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq 1}}^{b} 1 + \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq 1}}^{A-1} 1 + \mathcal{O}(1).$$

Proof. a) Since we prove the result with the additional $\mathcal{O}(1)$-term it suffices to prove the result for all a, b which both are larger than A and B.

We have $u^{lB+x} d^{lA+y} = u^x d^y$ for all integers l and all x and y. Hence, every value $u^x d^y$ with $0 \leq x \leq a$ and $0 \leq y \leq b$ equals $u^{x'} d^{y'}$ with some x' with $0 \leq x' \leq B-1$ and y' with $0 \leq y' \leq b$ or with some y' with $0 \leq y' \leq A-1$ and x' with $0 \leq x' \leq a$. Hence $M(a,b) \leq \overset{\circledast}{\sum} + \mathcal{O}(1)$.

On the other hand any two values $u^x d^y$ and $u^{x'} d^{y'}$ with $0 \leq x, x' \leq B-1$and $0 \leq y, y' \leq b$ or with $0 \leq x, x' \leq a$ and $0 \leq y, y' \leq A-1$ are different unless $(x,y) = (x', y')$, since $\frac{u^x}{e^y} = \frac{u^{x'}}{e^{y'}}$ implies $(x'-x)A = (y'-y)B$. Hence, $x' - x = lB$and $y' - y = lA$ for some integer l. If both x', x are in $[0, B-1]$ or if both $y', y$are in $[0, A-1]$, then $l = 0$ and $(x,y) = (x', y')$. If $l \neq 0$ and $x \in [0, B-1]$ and $y' \in [0, A-1]$, then $x' = x + lB$ and $y = y' - lA$, hence x' or y is negative, a contradiction. Hence $\overset{\circledast}{\sum} \leq M(a,b) + \mathcal{O}(1)$, and the result follows. (Note that the twice counted x and y in $\overset{\circledast}{\sum}$ are absorbed by the $\mathcal{O}(1)$-term.)

Part b) is shown quite analogously.

THEOREM 1. *If u,d are such that $\alpha := -\frac{\log u}{\log d} =: \frac{A}{B}$ is rational, then for the complexity Q of the backwardation algorithm we have*

$$Q = \frac{AB}{6(A+B)} n^3 + \mathcal{O}(n^2)$$

REMARK. *Note that the result coincides with the result of Example 1, by inserting $A = B = 1$.*

Proof. By Lemma 1 it remains to show that

$$\sum_{\substack{a=0}}^{n} \sum_{\substack{b=0 \\ \frac{u^a}{e^b} \geq 1}}^{n-a} \sum_{x=0}^{B-1} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq \frac{u^a}{e^b}}}^{b} 1 + \sum_{a=0}^{n} \sum_{\substack{b=0 \\ \frac{u^a}{e^b} \geq 1}}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq \frac{u^a}{e^b}}}^{A-1} 1 +$$
$$\sum_{a=0}^{n} \sum_{\substack{b=0 \\ \frac{u^a}{e^b} < 1}}^{n-a} \sum_{x=0}^{B-1} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq 1}}^{b} 1 + \sum_{a=0}^{n} \sum_{\substack{b=0 \\ \frac{u^a}{e^b} < 1}}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \geq 1}}^{A-1} 1 =:$$
$$=: \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4 = \frac{AB}{6(A+B)} n^3 + \mathcal{O}(n^2).$$

Consider $\sum_1$: For given a, b and x with $0 \le x \le B-1$ the conditions $\frac{u^x}{e^y} \ge \frac{u^a}{e^b}$ and $\frac{u^a}{e^b} \ge 1$ imply $\frac{u^x}{e^y} \ge 1$, hence $y \le x\frac{A}{B} < A$ and therefore $\sum_1 = O(n^2)$.

Consider $\sum_2$: Here for given a, b and x with $0 \le x \le B-1$ the condition $\frac{u^x}{e^y} \ge 1$ implies $y \le x\frac{A}{B} < A$ and so also $\sum_3 = O(n^2)$.

It remains to study $\sum_2$ and $\sum_4$.

We start with $\sum_2$: The conditions $\frac{u^a}{e^b} \ge 1$ and $\frac{u^x}{e^y} \ge \frac{u^a}{e^b}$ are equivalent with $b \le \left\lfloor a\frac{A}{B}\right\rfloor$ and $y \le b - (a-x)\frac{A}{B}$. Hence

$$\sum_2 = \sum_{a=0}^{n} \sum_{b=0}^{min\left(n-a,\left\lfloor a\frac{A}{B}\right\rfloor\right)} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ y<b-(a-x)\frac{A}{B}}}^{A-1} 1.$$

The innermost sum has the value A if $b-(a-x)\frac{A}{B} > A-1$, i.e., if $x > a+B-(b+1)\frac{A}{B}$ and it has the value $\left\lfloor b-(a-x)\frac{A}{B}\right\rfloor + 1$ if $0 < b-(a-x)\frac{A}{B} \le A-1$, i.e., if $a-\frac{B}{A}b < x \le B-(b+1)\frac{B}{A} + a$. Therefore

$$\sum_2 = \sum_{a=0}^{n} \sum_{b=0}^{min\left(n-a,\left\lfloor a\frac{A}{B}\right\rfloor\right)} \sum_{x=\left\lfloor a-\frac{B}{A}b\right\rfloor+1}^{\left\lfloor B-(b+1)\frac{B}{A}+a\right\rfloor} \left(\left\lfloor b-(a-x)\frac{A}{B}\right\rfloor + 1\right) +$$
$$+ \sum_{a=0}^{n} \sum_{b=0}^{min\left(n-a,\left\lfloor a\frac{A}{B}\right\rfloor\right)} \sum_{x=max\left(0,\left\lfloor a+B-(b+1)\frac{B}{A}\right\rfloor+1\right)}^{a} A =: \sum_{21} + \sum_{22} .$$

In $\sum_{21}$ the summand always is at most A and the number of x in $\sum_x$ is at most $\left\lfloor B-(b+1)\frac{B}{A}+a\right\rfloor - \left\lfloor a-\frac{B}{A}b\right\rfloor \le B+1$, hence $\sum_{21} = O(n^2)$, and $\sum_2 = \sum_{22} + O(n^2)$.

Further

$$\begin{aligned}
\sum_{22} &= \sum_{a=0}^{n} \sum_{b=0}^{min\left(n-a,\left\lfloor a\frac{A}{B}\right\rfloor\right)} \sum_{x=max\left(0,\left\lfloor a-b\frac{B}{A}\right\rfloor+1\right)}^{a} A + O(n^2) \\
&= \sum_{a=0}^{n} \sum_{b=0}^{min\left(n-a,\left\lfloor a\frac{A}{B}\right\rfloor\right)} bB + O(n^2) \\
&= \sum_{a=0}^{\left\lfloor\frac{n}{1+A/B}\right\rfloor} \sum_{b=0}^{\left\lfloor a\frac{A}{B}\right\rfloor} bB + \sum_{a=\left\lfloor\frac{n}{1+A/B}\right\rfloor+1}^{n} \sum_{b=0}^{n-a} bB + O(n^2) \\
&= \frac{A^2B}{6(A+B)^2} n^3 + O(n^2).
\end{aligned}$$

The reader may trust us, that by quite analogous elementary calculation we obtain

$$\sum_4 = \frac{B^2A}{6(A+B)^2} n^3 + O(n^2),$$

and the result follows

4. Some tools from uniform distribution theory

Let us turn now to the case of irrational α. To handle this case we will need two well-known tools from uniform distribution theory, namely the Koksma-Hlawka inequality and an estimate for the discrepancy of the Kronecker-sequence $\{n\alpha\}$ for n = 1, 2, 3, ... These tools will lead to Lemma 2, which will be needed for the proof of Theorem 2 in Chapter 5.

For an irrational number β with infinite continued fraction expansion $\beta := [b_0, b_1, b_2, \dots]$ and resulting best approximation denominators $q_0 \leq q_1 < q_2 < \cdots$, and for an arbitrary positive integer n let s be such that $q_s \leq n < q_{s+1}$. Then we define $KB_n(\beta) := \frac{n}{q_s} + \sum_{i=1}^{s} b_i$.

Note that $KB_n(\beta) \leq n$ always and that for all β irrational we have $KB_n(\beta) = o(n)$.

Moreover, for almost all irrationals β it is $KB_n(\beta) = o((\log n)^{1+\epsilon})$ for all $\epsilon > 0$ (see [4]).

LEMMA 2. *There is an absolute constant* c *such that*

a) $\left|\sum_{l=1}^{n} \{l\beta\}\left(\frac{l}{n}\right)^2 - \frac{1}{6}\right| \leq cKB_n(\beta)$ *for all* n*;*

b) $\left|\sum_{l=1}^{n} \{l\beta\}\left(1 - \frac{l}{n+\delta}\right)^2 - \frac{1}{6}\right| \leq cKB_n(\beta)$ *for all* n *and all* δ *with* $0 \leq \delta < 1$*;*

c) $\left|\sum_{l=1}^{n} \{l\beta\} - \frac{1}{2}\right| \leq cKB_n(\beta)$.

Proof. a) This follows immediately by applying the two-dimensional Koksma-Hlawka-inequality (see [[5], Theorem 5.5 in Part 2]) to the function $f: [0,1]^2 \to \mathbb{R}$ with $f(x,y) := xy^2$, which is of bounded variation V in the sense of Hardy and Krause, and which satisfies $\int_0^1 \int_0^1 f(x,y)d\,x\,dy = 1/6$, i.e.,

$$\left|\int_0^1 \int_0^1 f(x,y)\,dxdy - \frac{1}{n}\sum_{l=1}^{n} f\left(\{l\beta\}, \frac{l}{n}\right)\right| \leq V \cdot D_n,$$

where D_n denotes the discrepancy of the point set $\left(\{l\beta\}, \frac{l}{n}\right)$ $l = 1,2,\dots,n$ in $[0,1]^2$. And finally, it follows from the fact, that for the discrepancy D_n we have $D_n \leq c' \cdot \frac{KB_n(\beta)}{n}$ with an absolute constant c'. This is an immediate consequence of formula (3.18) in the proof of Theorem 3.4 in Part 2 of [5] concerning the discrepancy of the Kronecker-sequence $\{l\beta\}$ for $l = 1, 2, 3, \dots$ and by Example 2.2 in Part 2 of [5].

a) For the point set $\left(\{l\beta\}, 1 - \frac{l}{n+\delta}\right)$ for $l = 0, \dots, n$ we have the same discrepancy estimate as used for $\left(\{l\beta\}, \frac{l}{n}\right)$, $l = 1, \dots, n$, since it is just a slightly perturbed and mirrored version of the second point set, and so also this result follows.

b) This follows immediately from the one-dimensional Koksma-Hlawka in-equality for $f(x) = x$ and the discrepancy estimate for Kronecker sequences cited above.

For the proof of Theorem 2 we also will use the following simple fact:

LEMMA 3. *For every α irrational there is a constant $c = c(\alpha)$ such that for all n we have*

$$KB_n\left(\frac{1}{\alpha}\right) \leq c \cdot KB_n(\alpha)$$

Proof. Let first $\alpha < 1, \alpha := [0, b_1, b_2, \ldots]$ and $q_s \leq n < q_s + 1$. By definition $KB_n(\alpha) = b_1 + b_2 + \cdots + b_s + \frac{n}{q_s}$. We have $\frac{1}{\alpha} = [b_1, b_2, b_3, \ldots]$. Let $Q_0 \leq Q_1 < Q_2 < \cdots$ be the best approximation denominations of $\frac{1}{\alpha}$, i.e., $Q_{i+1} = b_{i+2}Q_i + Q_{i-1}$, then

$$\begin{aligned}\begin{pmatrix} p_{s-1} & p_s \\ q_{s-1} & q_s \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & b_s \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}\begin{pmatrix} P_{s-2} & P_{s-1} \\ Q_{s-2} & Q_{s-1} \end{pmatrix},\end{aligned}$$

hence $Q_{s-1} < q_s$ and, more detailed,

$$q_s = P_{s-1} + b_1 Q_{s-1} \leq (\alpha + 1)Q_{s-1} + b_1 Q_{s-1} \leq \left(\alpha + 1 + \frac{1}{\alpha}\right)Q_{s-1} =: \kappa Q_{s-1}$$

With $\kappa := \alpha + 1 + \frac{1}{\alpha}$

We have $KB_n\left(\frac{1}{\alpha}\right) = b_2 + b_3 + \cdots + b_s + \cdots + b_{s+v+1} + \frac{n}{Q_{s+v}}$ if $v \geq -1$ is such that $Q_{s+v} \leq n < Q_{s+v+1}$. Further

$$\frac{n}{Q_{s+v}} \frac{Q_{s+v}}{Q_{s+v-1}} \cdots \frac{Q_s}{Q_{s-1}} = \frac{n}{Q_{s-1}} \leq \frac{n\kappa}{q_s}.$$

Assume now, that v is odd and ≥ 1, say $v = 2t + 1$ and let $x_i := \frac{Q_{s+2i}}{Q_{s+2i-1}} \cdot \frac{Q_{s+2i+1}}{Q_{s+2i}}$. Note that $x_i = \frac{Q_{s+2i+1}}{Q_{s+2i-1}} \geq 2$ for all i, hence $x_0 x_1 \cdots x_t \geq x_0 + x_1 + \cdots + x_t$. Further

$$x_i = \frac{Q_{s+2i}}{Q_{s+2i-1}} \frac{Q_{s+2i+1}}{Q_{s+2i}} \geq \frac{1}{2}\left(\frac{Q_{s+2i}}{Q_{s+2i-1}} + \frac{Q_{s+2i+1}}{Q_{s+2i}}\right) \geq \frac{1}{2}(b_{s+2i+1} + b_{s+2i+1}),$$

Hence

$$\begin{aligned}\frac{n\kappa}{q_s} &\geq \frac{n}{2Q_{s+v}}(b_{s+v+1} + b_{s+v} + \cdots + b_{s+1}) \\ &\geq \frac{1}{5}\left(\frac{n}{Q_{s+v}} + b_{s+v+1} + b_{s+v} + \cdots + b_{s+1}\right)\end{aligned}$$

(note that $\frac{n}{2Q_{s+v}} \geq \frac{1}{2}$ and $b_{s+v+1} + b_{s+v} + \cdots + b_{s-1} \geq 2$).

Altogether we have

$$\begin{aligned}KB_n\left(\frac{1}{\alpha}\right) &= b_2 + \cdots + b_s + b_{s+1} + \cdots + b_{s+v+1} + \frac{n}{Q_{s+v}} \\ &\leq 5\kappa\frac{n}{q_s} + b_2 + b_3 + \cdots b_s \\ &\leq 5\kappa KB_n(\alpha).\end{aligned}$$

The cases $v = -1$, v even and $\alpha > 1$ are dealt with quite analogously with obvious adaptions and will not be carried out here in detail.

5. The complexity of the algorithm for irrational α

In the irrational case we will show:

THEOREM 2. *If u, d are such that* $\alpha := -\frac{\log u}{\log d}$ *is irrational, then for the complexity* Q *of the backwardation algorithm we have*

$$Q = \frac{\alpha}{24(1+\alpha)^2} n^4 + \frac{1}{12}\left(1 + \frac{\alpha}{(1+\alpha)^2}\right) n^3 + O(KB_n(\alpha) n^2),$$

where $KB_n(\alpha)$ *is as defined in Chapter 4.*

REMARK. *Note again that* $KB_n(\alpha) = o(n)$ *for all irrational* α. *Furthermore,* $KB_n(\alpha) = o((\log n)^{1+\epsilon})$ *for almost all* α *for all* $\epsilon > 0$.

Proof. Since for α irrational we have $\frac{u^x}{d^y} = \frac{u^{x'}}{d^{y'}}$ for some integers x, x', y, y' if and only if $(x, y) = (x', y')$, we just have to estimate the sum

$$Q = \sum_{\substack{a,b=0 \\ a+b \le n}}^{n} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \ge \max\left(1, \frac{u^a}{e^b}\right)}}^{b} 1.$$

We split the sum according to $\frac{u^a}{e^b} \ge 1$, or $\frac{u^a}{e^b} < 1$, i.e., according to $b \le a\alpha$, or $b > a\alpha$:

$$Q = \sum_{a=0}^{n} \sum_{\substack{b=0 \\ b \le a\alpha}}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \ge \frac{u^a}{e^b}}}^{b} 1 + \sum_{a=0}^{n} \sum_{\substack{b=0 \\ b > a\alpha}}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y} \ge 1}}^{b} 1 =: \sum\nolimits_1 + \sum\nolimits_2$$

Using the abbreviation $M := \left\lfloor \frac{n}{1+\alpha} \right\rfloor$ we have

$$\begin{aligned} \sum\nolimits_1 &= \sum_{a=0}^{M} \sum_{b=0}^{[a\alpha]} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ (b-y) \ge (a-x)\alpha}}^{b} 1 \\ &\quad + \sum_{a=M+1}^{n} \sum_{x=0}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ (b-y) \ge (a-x)\alpha}}^{b} 1 \\ &= \sum\nolimits_{11} + \sum\nolimits_{12} . \end{aligned}$$

We now simplify the inner double sum $\sum_x \sum_y \ldots$ in these expressions:

$$\sum_{x=0}^{a} \sum_{\substack{y=0 \\ (b-y) \ge (a-x)\alpha}}^{b} 1 = \sum_{x'=0}^{a} \sum_{\substack{y'=0 \\ y' \ge x'\alpha}}^{b} 1 = \sum_{y=0}^{b} \sum_{x=0}^{\left\lfloor \frac{y}{\alpha} \right\rfloor} 1 = \sum_{y=0}^{b} \left(\left\lfloor \frac{y}{\alpha} \right\rfloor + 1 \right).$$

Here the last-but-one equality holds since the conditions $b \le a\alpha$ made for $\sum_1$ and $y \ge x\alpha$ imply $x \le \frac{y}{\alpha} \le \frac{b}{\alpha} \le a$, hence x runs from 0 to $\left\lfloor \frac{y}{\alpha} \right\rfloor$. We get

$$\begin{aligned}\Sigma_{11} &= \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} \sum_{y=0}^{b} \left(\left\lfloor \frac{y}{\alpha} \right\rfloor + 1 \right) \\ &= \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} \sum_{y=0}^{b} \left(\frac{y}{\alpha} + 1 \right) - \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} \sum_{y=0}^{b} \left\{ \frac{y}{\alpha} \right\} =: \Sigma_{111} - \Sigma_{112} \, .\end{aligned}$$

We have

$$\begin{aligned}\Sigma_{111} &= \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} \left(\frac{1}{\alpha} \frac{b(b+1)}{2} + b + 1 \right) \\ &= \frac{1}{2\alpha} \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} b^2 + \left(\frac{1}{2\alpha} + 1 \right) \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} b + \mathcal{O}(n^2) \\ &= \frac{1}{2\alpha} \sum_{a=0}^{M} \frac{\lfloor a\alpha \rfloor (\lfloor a\alpha \rfloor + 1)(2\lfloor a\alpha \rfloor + 1)}{6} \\ &\quad + \left(\frac{1}{2\alpha} + 1 \right) \sum_{a=0}^{M} \frac{\lfloor a\alpha \rfloor (\lfloor a\alpha \rfloor + 1)}{2} + \mathcal{O}(n^2) \\ &= \frac{1}{6\alpha} \sum_{a=0}^{M} \lfloor a\alpha \rfloor^3 + \left(\frac{1}{2\alpha} + \frac{1}{2} \right) \sum_{a=0}^{M} \lfloor a\alpha \rfloor^2 + \mathcal{O}(n^2).\end{aligned}$$

Now we see that

$$\begin{aligned}\sum_{a=0}^{M} \lfloor a\alpha \rfloor^2 &= \sum_{a=0}^{M} (a\alpha - \{a\alpha\})^2 = \alpha^2 \sum_{a=0}^{M} a^2 + \mathcal{O}(n^2) \\ &= \alpha^2 \frac{M(M+1)(2M+1)}{6} + \mathcal{O}(n^2) = \frac{\alpha^2}{3} M^3 + \mathcal{O}(n^2)\end{aligned}$$

and

$$\begin{aligned}\sum_{a=0}^{M} \lfloor a\alpha \rfloor^3 &= \sum_{a=0}^{M} (a\alpha - \{a\alpha\})^3 \\ &= \alpha^3 \sum_{a=0}^{M} a^3 - 3\alpha^2 \sum_{a=0}^{M} a^2 \{a\alpha\} + \mathcal{O}(n^2) \\ &= \frac{\alpha^3}{4} M^2 (M+1)^2 - 3\alpha^2 M^3 \left(\frac{1}{M} \sum_{a=1}^{M} \left(\frac{a}{M} \right)^2 \{a\alpha\} - \frac{1}{6} \right) - \frac{\alpha^2}{2} M^3 \\ &= \frac{\alpha^3}{4} M^2 (M+1)^2 - \frac{\alpha^2}{2} M^3 - 3\alpha^2 M^3 E_1,\end{aligned}$$

with

$$E_1 := \frac{1}{M} \sum_{a=1}^{M} \left(\frac{a}{M} \right)^2 \{a\alpha\} - \frac{1}{6}.$$

Hence

$$\begin{aligned}\Sigma_{111} &= \frac{\alpha^2}{24} M^2 (M+1)^2 - \frac{\alpha}{12} M^3 - \frac{\alpha}{2} E_1 + \left(\frac{1}{2\alpha} + \frac{1}{2} \right) \frac{\alpha^2}{3} M^3 + \mathcal{O}(n^2) \\ &= \frac{\alpha^2}{24} M^4 + \left(\frac{\alpha^2}{4} + \frac{\alpha}{12} \right) M^3 - \frac{\alpha}{2} M^3 E_1 + \mathcal{O}(n^2).\end{aligned}$$

Further

$$\begin{aligned}
\Sigma_{112} &= \sum_{a=0}^{M} \sum_{b=0}^{\lfloor a\alpha \rfloor} \sum_{y=0}^{b} \left\{\frac{y}{\alpha}\right\} = \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \sum_{\substack{a=0\\ y\le \lfloor a\alpha \rfloor}}^{M} \sum_{b=y}^{\lfloor a\alpha \rfloor} \\
&= \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \sum_{a=0}^{M} \left([a\alpha] - y + 1\right) \\
&= \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \sum_{l=1}^{M-\left\lfloor \frac{y}{\alpha} \right\rfloor} l\alpha + \mathcal{O}(n^2) \\
&= \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \frac{\alpha}{2}\left(M - \frac{y}{\alpha}\right)^2 + \mathcal{O}(n^2) \\
&= \frac{\alpha}{2} M^2 \lfloor M\alpha \rfloor \left(\frac{1}{\lfloor M\alpha \rfloor} \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \left(1 - \frac{y}{M\alpha}\right)^2\right) + \mathcal{O}(n^2) \\
&= \frac{\alpha^2 M^3}{12} + \frac{\alpha^2 M^3}{2} E_2 + \mathcal{O}(n^2),
\end{aligned}$$

where

$$E_2 := \frac{1}{\lfloor M\alpha \rfloor} \sum_{y=1}^{\lfloor M\alpha \rfloor} \left\{\frac{y}{\alpha}\right\} \left(1 - \frac{y}{M\alpha}\right)^2 - \frac{1}{6}.$$

Summing up, we get

$$\begin{aligned}
\Sigma_{11} &= \Sigma_{111} - \Sigma_{112} \\
&= \frac{\alpha^2}{24} M^4 + \left(\frac{\alpha^2}{6} + \frac{\alpha}{12}\right) M^3 - \frac{\alpha}{2} M^3 E_1 - \frac{\alpha^2}{2} M^3 E_2 + \mathcal{O}(n^2).
\end{aligned}$$

We turn now to Σ_{12}:

$$\begin{aligned}
\Sigma_{12} &= \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0\\ (b-y)\ge (a-x)\alpha}}^{b} 1 \\
&= \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \sum_{y=0}^{b} \left(\left\lfloor \frac{y}{\alpha} \right\rfloor + 1\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \sum_{y=0}^{b} \left(\frac{y}{\alpha} + 1\right) - \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \sum_{y=0}^{b} \left\{\frac{y}{\alpha}\right\} \\
&=: \Sigma_{121} - \Sigma_{122} .
\end{aligned}$$

We have

$$\begin{aligned}
\Sigma_{121} &= \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \left(\frac{1}{\alpha}\frac{b(b+1)}{2} + b + 1\right) \\
&= \frac{1}{2\alpha} \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} b^2 + \left(\frac{1}{2\alpha} + 1\right) \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} b + \mathcal{O}(n^2) \\
&= \frac{1}{24\alpha} (n-M)^2\left((n-M)^2 - 1\right) \\
&\quad + \left(\frac{1}{12\alpha} + \frac{1}{6}\right) (n-M)\left((n-M)^2 - 1\right) + \mathcal{O}(n^2) \\
&= \frac{1}{24\alpha} (n-m)^4 + \left(\frac{1}{12\alpha} + \frac{1}{6}\right) (n-M)^3 + \mathcal{O}(n^2).
\end{aligned}$$

Further

$$\begin{aligned}\Sigma_{122} &= \sum_{a=M+1}^{n} \sum_{b=0}^{n-a} \sum_{y=0}^{b} \left\{\frac{y}{\alpha}\right\} \\ &= \sum_{y=1}^{n-M-1} \left\{\frac{y}{\alpha}\right\} \sum_{a=M+1}^{n-y} \sum_{b=y}^{n-a} 1 \\ &= \sum_{y=1}^{n-M-1} \left\{\frac{y}{\alpha}\right\} \sum_{a=M+1}^{n-y} (n-a-y+1) \\ &= \sum_{y=1}^{n-m-1} \left\{\frac{y}{\alpha}\right\} \frac{(n-M-y)(n-M-y+1)}{2} \\ &= \frac{1}{2}\sum_{y=1}^{n-M} \left\{\frac{y}{\alpha}\right\} (n-M-y)^2 + \mathcal{O}(n^2) \\ &= \frac{(n-M)^3}{2}\left(\frac{1}{n-M}\sum_{y=1}^{n-M} \left\{\frac{y}{\alpha}\right\}\left(1-\frac{y}{n-M}\right)^2\right) + \mathcal{O}(n^2) \\ &= \frac{(n-M)^3}{12} + \frac{(n-M)^3}{2} E_3,\end{aligned}$$

where

$$E_3 := \frac{1}{n-M}\sum_{y=1}^{n-M} \left\{\frac{y}{\alpha}\right\}\left(1-\frac{y}{n-M}\right)^2 - \frac{1}{6}.$$

Summing up we get

$$\begin{aligned}\Sigma_{12} &= \Sigma_{121} - \Sigma_{122} \\ &= \frac{1}{24\alpha}(n-M)^4 + \left(\frac{1}{12\alpha} + \frac{1}{12}\right)(n-M)^3 - \frac{(n-M)^3}{2} E_3 + \mathcal{O}(n^2).\end{aligned}$$

Finally we have to deal with Σ_2:

$$\begin{aligned}\Sigma_2 &= \sum_{a=0}^{n} \sum_{\substack{b=0 \\ b>a\alpha}}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ \frac{u^x}{e^y}\geq 1}}^{b} 1 \\ &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} \sum_{\substack{y=0 \\ y<x\alpha}}^{b} 1 \\ &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} \sum_{y=0}^{\lfloor x\alpha\rfloor} 1 \\ &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} (\lfloor x\alpha\rfloor + 1) \\ &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} (x\alpha+1) - \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} \{x\alpha\} \\ &= \Sigma_{21} - \Sigma_{22}\ .\end{aligned}$$

Now

$$\begin{aligned}\Sigma_{21} &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \left(\alpha\frac{a(a+1)}{2} + (a+1)\right) \\ &= \frac{\alpha}{2}\sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} a^2 + \left(\frac{\alpha}{2}+1\right)\sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} a + \mathcal{O}(n^2) \\ &= \frac{\alpha}{2}\sum_{a=0}^{M} a^2(n-a-\lfloor a\alpha\rfloor) + \left(\frac{\alpha}{2}+1\right)\sum_{a=0}^{M} a(n-a-\lfloor a\alpha\rfloor) + \mathcal{O}(n^2)\end{aligned}$$

$$\begin{aligned}&= \sum_{a=0}^{M} \left(\frac{\alpha}{2}a^2 + \left(\frac{\alpha}{2}+1\right)a\right)(n-a-a\alpha) + \frac{\alpha}{2}\sum_{a=0}^{M} a^2\{a\alpha\} + \mathcal{O}(n^2) \\ &= \frac{\alpha}{6}M^3 n - M^4\left(\frac{\alpha}{8}+\frac{\alpha^2}{8}\right) - M^3\left(\frac{5\alpha^2}{12}+\frac{3\alpha}{4}+\frac{1}{3}\right) \\ &\quad + \left(\frac{\alpha}{2}+\frac{1}{2}\right)M^2 n - \frac{\alpha}{2}\sum_{a=0}^{M} a^2\{a\alpha\} + \mathcal{O}(n^2).\end{aligned}$$

Further

$$\sum\nolimits_{21} = \frac{\alpha}{6} M^3 n - M^4 \left(\frac{\alpha}{8} + \frac{\alpha^2}{8}\right) - M^3 \left(\frac{5\alpha^2}{12} + \frac{2\alpha}{3} + \frac{1}{3}\right) + \left(\frac{\alpha}{2} + \frac{1}{2}\right) M^2 n + \frac{\alpha}{2} M^3 E_4 + \mathcal{O}(n^2),$$

where

$$E_4 := \frac{1}{M} \sum_{a=0}^{M} \left(\frac{a}{M}\right)^2 \{a\alpha\} - \frac{1}{6}.$$

It remains to estimate $\sum_{22}$:

$$\begin{aligned}\sum\nolimits_{22} &= \sum_{a=0}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} \sum_{x=0}^{a} \{x\alpha\} \\ &= \sum_{x=0}^{M} \{x\alpha\} \sum_{a=x}^{M} \sum_{b=\lfloor a\alpha\rfloor+1}^{n-a} 1 \\ &= \sum_{x=0}^{M} \{x\alpha\} \sum_{a=x}^{M} (n - a(\alpha+1)) + \mathcal{O}(n^2) \\ &= \sum_{x=0}^{M} \{x\alpha\} \left(-\frac{M^2}{2} + Mn - \frac{M^2\alpha}{2}\right) + \mathcal{O}(n^2) \\ &= E_5 \left(-\frac{M^3}{2} + M^2 n - \frac{M^3\alpha}{2}\right) - \frac{M^3}{4} + \frac{M^2 n}{2} - \frac{M^3\alpha}{4} + \mathcal{O}(n^2),\end{aligned}$$

Where $E_5 := \frac{1}{M}\sum_{x=0}^{M} \{x\alpha\} - \frac{1}{2}$.

Now we can sum up $Q = \sum_{11} + \sum_{12} + \sum_{21} - \sum_{22}$ and insert $M := \frac{n}{1+\alpha}\tau$ with some τ with $0 \le \tau < 1$ and we obtain

$$Q = \frac{n^4}{24}\frac{\alpha}{(1+\alpha)^2} + \frac{n^3}{12}\left(1 + \frac{\alpha}{(1+\alpha)^2}\right) - \frac{\alpha}{2} M^3 E_1 - \frac{\alpha^2}{12} M^3 E_2 - \frac{(n-M)^3}{2} E_3 + \frac{\alpha}{2} M^3 E_4 + \left(M^2 n - \frac{M^3}{2} - \frac{M^3\alpha}{2}\right) E_5 + \mathcal{O}(n^2),$$

and from Lemma 2 and Lemma 3 we obtain the result

$$Q = \frac{n^4}{24}\frac{\alpha}{(1+\alpha)^2} + \frac{n^3}{12}\left(1 + \frac{\alpha}{(1+\alpha)^2}\right) + \mathcal{O}(n^2 K B_n(\alpha)).$$

REFERENCES

[1] Gaudenzi, M., Lepellere, M. A., Zanette, A. The Singular Point Method for Pricing Path-Dependent Options. Journal of Computational Finance 14-1, 29-56. 2010.

[2] Hull, J. Options, Futures and other Derivatives (9th Edition). Pearson 2014

[3] Jiang, L. and Dai, M. Convergence of Binomial Tree Methods for European/American path dependent options. SIAM J. Numer. Anal. 42, Nr 3. pp 1094-1109, 2004.

[4] Khintchine, A. Einige SÅNatze ÅNuber KettenbrÅNuche mit Anwendungen auf die Theorie der Diophantischen Approximationen. Math.Ann. 92, 115-125. 1924.

[5] Kuipers, L. and Niederreiter, H. Uniform Distribution of Sequences. Wiley 1974

[6] Larcher, G. The Art of Qantitative Finance, Volume 2., Springer 2023

[7] Wagner, M. Pricing of American Lookback Options using Linear Programming. Thesis and Dissertations 206, University of Wisconsin, 2012.

Received October 24, 2024
Accepted April 29, 2025

G. Larcher

Institute of Financial Mathematics and Applied Number Theory Johannes Kepler University Linz
Austria

S. Pomberger
Institute of Financial Mathematics and Applied Number Theory Johannes Kepler University Linz
Austria
E-mail: stefan.pomberger@stanaholdingag.ch