

WEYL'S UNIFORM DISTRIBUTION UNDER PERIODIC PERTURBATION

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ABSTRACT. We examine the uniform distribution theory of H. Weyl when there is a periodic perturbation present. As opposed to the classical setting, in this case the conditions for $(\bmod 1)$ density and $(\bmod 1)$ uniform distribution turn out to be different.

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1. Introduction

In connection with asymptotics of orthogonal polynomials Peter Yuditskii asked if the numbers $n\alpha - \cos(n\pi\alpha + \phi_0)$, $n = 1, 2, \dots$, where α is irrational and ϕ_0 is a given number, are dense modulo 1 (in short $(\bmod 1)$). Recall that if α is irrational, then the sequence $n\alpha$, $n = 1, 2, \dots$, is dense in $[0, 1) (\bmod 1)$, so the problem asks if the same remains true if we perturb this sequence in the fashion described.

This paper has emerged from this question, and will examine classical uniform distribution theory when there is a periodic (or almost periodic) perturbation present.

We shall arrive at the most general results through simpler cases because those simpler cases are interesting in themselves and because the proofs will be more transparent by not repeating simpler arguments as we proceed to higher generality.

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Let us begin with a simple result with an elementary proof that answers Yuditskii's question.

PROPOSITION 1.1. *If f is a continuous periodic function and α is irrational, then the sequence $\{n\alpha + f(n)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$.*

Thus, mod 1 density is always the case, but we shall see later that $(\text{mod } 1)$ uniform distribution may not be true. This is in contrast with the classical setting, where the conditions for density and uniform distribution are mostly the same.

We shall have much more general density results, but for a start we give an elementary proof of Proposition 1.1.

Proof. Let $\varepsilon > 0$ be small, and let L be a large integer. Choose $\delta < 1/2$ such that $|f(u) - f(v)| < \varepsilon$ if $|u - v| < \delta$, and let $|f| \leq M$, M integer. If β is the/a period of f , then choose a k such that $k(\alpha/L) \in [-\varepsilon/L, \varepsilon/L] \pmod{1}$ and $k(1/\beta) \in [-\delta/\beta, \delta/\beta] \pmod{1}$ (possible by Dirichlet's simultaneous approximation theorem, see [2, Theorem 201]). Then $k\alpha \in [-\varepsilon, \varepsilon] \pmod{L}$ and $k \in [-\delta, \delta] \pmod{\beta}$.

Assume, without loss of generality, that $k\alpha \in [0, \varepsilon] \pmod{L}$, say $k\alpha = k_1L + \alpha_1$, where k_1 is an integer and $\alpha_1 \in (0, \varepsilon]$ (since α is irrational, α_1 cannot be 0). Consider the numbers

$$x_m = mk\alpha, \quad 1 \leq m \leq N,$$

where $N = [L/\alpha_1]$. We have $x_m = m(k\alpha - k_1L) = m\alpha_1 \pmod{L}$, and these last numbers $m\alpha_1$ are equidistant on the interval $[0, L]$ with neighboring distance $\alpha_1 \leq \varepsilon$. At the same time

$$|f(k(m+1)) - f(km)| = |f(km + (k \bmod \beta)) - f(km)| \leq \varepsilon$$

because of the choice of δ and because of $(k \bmod \beta) \in [-\delta, \delta]$, where $(k \bmod \beta)$ denotes the number k' of smallest absolute value for which $k = k' \pmod{\beta}$.

Let $M_1 = [M/\alpha_1] + 1$. Since

$$M \leq M_1\alpha_1 \leq M + \alpha_1 < M + 1 \tag{1.1}$$

and

$$L - 2 - M < L - \alpha_1 - (M + \alpha_1) < (N - M_1)\alpha_1 < L - M, \tag{1.2}$$

it follows that if $y_m = x_m + f(km) \pmod{L}$, $y_m \in [0, L)$, then the sequence $\{y_m\}_{m=M_1}^{N-M_1}$ (which is the same as $\{m\alpha_1 + f(km)\}_{m=M_1}^{N-M_1}$ in view of (1.1), (1.2) and $|f| \leq M$) starts from a positive number $< 2M + 1$ and ends with a number $> L - 2M - 2$, and the distance in between consecutive terms is at most 2ε . Therefore, for every $y \in [2M + 1, L - 2M - 2]$ there is a term y_m in the sequence with $|y - y_m| \leq 2\varepsilon$, and for $L > 4M + 5$ (which guarantees that $[2M + 1, L - 2M - 2]$

contains two consecutive integers) this shows that to any point $y' \in [0, 1)$ there is a y_m closer than $2\varepsilon \pmod{1}$, and $\pmod{1}$ such a y_m equals $km\alpha + f(km)$. $\square$

To illustrate that there is room for generalization, let us consider almost periodic functions. Bohr's almost periodic (real) functions (sometimes called uniformly almost periodic functions) coincide with the functions which can be uniformly approximated on $\mathbf{R}$ by generalized (real) trigonometric polynomials of the form $\sum c_j e^{i\lambda_j x}$, see [1, Section 84]¹. Proposition 1.1 is the special case of the following result.

PROPOSITION 1.2. *If f is almost periodic and α is irrational, then the sequence $\{n\alpha + f(n)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$.*

Proof. First of all notice that the proof of Proposition 1.1 easily implies the following: if $f_1, \dots, f_\nu$ are continuous periodic functions and α is irrational, then the sequence $\{n\alpha + f_1(n) + \dots + f_\nu(n)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$. Indeed, let $\beta_1, \dots, \beta_\nu$ be the periods of $f_1, \dots, f_\nu$, respectively. As before, let $\varepsilon > 0$ be given, and let L be again a large integer. Choose δ such that $|f_\tau(u) - f_\tau(v)| < \varepsilon/\nu$ for all τ if $|u - v| < \delta$, and let $\sum_\tau |f_\tau| \leq M$, $1 \leq \tau \leq \nu$, M integer. Dirichlet's simultaneous approximation theorem can again be applied and we can choose a k such that $k(\alpha/L) \in [-\varepsilon/L, \varepsilon/L] \pmod{1}$ and $k(1/\beta_\tau) \in [-\delta/\beta_\tau, \delta/\beta_\tau] \pmod{1}$ for all τ . The rest of the proof proceeds as before with minor modifications.

In particular, it follows that if $T = \sum_{\tau=1}^{\nu} (a_\tau \cos(\lambda_\tau x) + b_\tau \sin(\lambda_\tau x))$ is a generalized trigonometric polynomial, then the sequence $\{n\alpha + T(n)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$. But every almost periodic function is the uniform limit of such generalized trigonometric polynomials, and the claim in the theorem follows: if $y \in (0, 1)$ and $\varepsilon > 0$ are given, then let $\eta = \min\{\varepsilon, y, 1 - y\}$, choose a T with $|f - T| < \eta/2$ and then an n with² $|y - \{n\alpha + T(n)\}| < \eta/2$, from which we obtain $|y - \{n\alpha + f(n)\}| < \eta < \varepsilon$. $\square$

2. Preliminaries, Weyl's theorems

Let $\{\cdot\}$ denote fractional part. There are two slightly different ways to talk about $\pmod{1}$ values of a sequence x_n , $n = 1, 2, \dots$, namely, one can look

¹The original definition of almost periodicity is as follows: a continuous real function f is said to be almost periodic if for every $\varepsilon > 0$ there is an $L(\varepsilon) > 0$ such for in every subinterval I of $\mathbf{R}$ of length $\geq L(\varepsilon)$ there is a $t \in I$ for which $|f(x+t) - f(x)| \leq \varepsilon$ for all $x \in \mathbf{R}$. That the space of almost periodic functions coincides with the uniform closure of generalized trigonometric polynomials was one of Bohr's main theorems.

² $\{\cdot\}$ denotes fractional part.

at the sequence $\{x_n\}$, $n = 1, 2, \dots$, of the fractional parts as elements of $[0, 1)$ or as elements of the torus $\mathbf{R}/\mathbf{Z}$ (i.e., when we identify 0 and 1 in $[0, 1]$). From the point of view of (mod 1) density or (mod 1) uniform distribution it does not matter which view we take. In our discussion it will be more convenient to work on $[0, 1)$ and not on the torus because the proofs are easier this way and because the perturbing function f may not be conveniently reduced to a function on the torus. We shall briefly discuss in Section 6 one question when the torus vs. $[0, 1)$ makes a difference.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. This sequence is said to be uniformly distributed on $[0, 1)$ (mod 1) if for all $0 \leq a < b < 1$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq k \leq n \mid \{x_k\} \in [a, b]\} = b - a.$$

This is the same that for every continuous function f on $[0, 1]$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\{x_k\}) = \int_0^1 f.$$

In a similar fashion, if $\{X_n\}_{n=1}^{\infty}$ is a sequence of vectors from $\mathbf{R}^d$, then its (mod 1) uniform distribution (on $[0, 1)^d$) means that for all $0 \leq a_i < b_i < 1$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \# \left\{ 1 \leq k \leq n \mid \{X_k\} \in \prod_{i=1}^d [a_i, b_i] \right\} = \prod_{i=1}^d (b_i - a_i),$$

which is the same that for every continuous function f on $[0, 1]^d$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(\{x_k\}) = \int_{[0, 1]^d} f.$$

Weyl's first theorem [5, Satz 1] ([3, Theorem 2.1]) on uniform distribution asserts that $\{x_n\}_{n=1}^{\infty}$ is uniformly distributed (mod 1) on $[0, 1)$ if and only if for all integers $a \neq 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{i2\pi a x_k} = 0.$$

More generally, if

$$\{(x_{1,n}, \dots, x_{d,n})\}_{n=1}^{\infty} \tag{2.1}$$

is a sequence from $\mathbf{R}^d$, then this sequence is uniformly distributed (mod 1) if and only if for all integer d -tuples $(a_1, \dots, a_d)$ with $\sum a_i^2 \neq 0$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{i2\pi(a_1 x_{1,k} + \dots + a_d x_{d,k})} = 0,$$

see [5, Satz 3] ([3, Theorem 6.2]).

In particular, (2.1) is uniformly distributed on $[0, 1)^d$ if and only if every non-trivial integer combination sequence $y_n = a_1x_{1,n} + \cdots + a_dx_{d,n}$, $n = 1, 2, \dots$, is uniformly distributed on $[0, 1)$. This is not true if we talk about density ((2.1) may not be dense in $[0, 1)^d$ even though every $\{y_n\}_{n=1}^\infty$ is dense in $[0, 1)$), but we shall see that for polynomial sequences density and uniform distribution are the same.

Let $P_1, \dots, P_d$ be real polynomials. It is clear that from the point of view of the (mod 1) density or (mod 1) uniform distribution of the sequence

$$(P_1(n), \dots, P_d(n)), \quad n = 1, 2, \dots,$$

the constant terms in the polynomials are irrelevant, therefore in what follows we shall always assume that the polynomials are without constant terms.

For polynomial sequences Weyl proved in [5, Satz 9] ([3, Theorem 3.2]) that if P (without a constant term) has at least one irrational coefficient, then $\{P(n)\}_{n=1}^\infty$ is uniformly distributed on $[0, 1)$ (mod 1). On the other hand, if all coefficients of P are rational, then it is clear that $\{P(n)\}_{n=1}^\infty$ has only finitely many different terms (mod 1), so this sequence is not dense in $[0, 1)$ (mod 1).

To deal with the vector case we introduce the following definition.

DEFINITION 2.1. We call polynomials $P_1, \dots, P_d$ (without constant terms) $\mathbf{Q}$ -independent, if no non-trivial rational combination $r_1P_1 + \cdots + r_dP_d$ is a polynomial with rational coefficients.

Clearly, this is the same that no non-trivial linear combination of $P_1, \dots, P_d$ with integer coefficients is a polynomial with integer coefficients.

Combining Weyl's theorems discussed so far we can see that if $P_1, \dots, P_d$ are polynomials without constant terms, then

$$\{(P_1(n), \dots, P_d(n))\}_{n=1}^\infty \tag{2.2}$$

is uniformly distributed in $[0, 1)^d$ (mod 1) precisely when $P_1, \dots, P_d$ are $\mathbf{Q}$ -independent, i.e., every non-trivial rational linear combination $\sum_j \alpha_j P_j$ has at least one irrational coefficient. Furthermore, the condition is the same for the (mod 1) density of the sequence (2.2) in $[0, 1)^d$. Indeed, if there is a non-trivial linear combination $r_1P_1 + \cdots + r_dP_d$ with integer coefficients which has integer coefficients, then $r_1\{P_1(n)\} + \cdots + r_d\{P_d(n)\}$ is an integer r lying in the interval $[-(|r_1| + \cdots + |r_d|), |r_1| + \cdots + |r_d|]$. Therefore, $(\{P_1(n)\}, \dots, \{P_d(n)\})$ lies in finitely many hyperplanes $r_1x_1 + \cdots + r_dx_d = r$ of $\mathbf{R}^d$, hence this sequence is not dense in $[0, 1)^d$.

We can specialize these results to the case when $P_j(x) = \alpha_j x$ with some given α_j and obtain that the sequence

$$\{(n\alpha_1, \dots, n\alpha_d)\}_{n=1}^{\infty} \quad (2.3)$$

of vectors is uniformly distributed (mod 1) on $[0, 1)^d$ if and only if it is dense (mod 1) in $[0, 1)^d$, and this happens precisely if the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent, i.e., no non-trivial rational linear combination of them is 0.

3. Density

In this section we shall deal with the problem discussed in the introduction in more generality. We shall see that periodic or almost periodic perturbations never prevent density (mod 1). The story will be different when we shall deal with uniform distribution in the next section.

The following proposition is equivalent to Proposition 1.2, but we present a formulation and a more streamlined proof for it that will be the basis of the polynomial and vector cases to be followed.

PROPOSITION 3.1. *If α is irrational and $f_1, \dots, f_\nu$ are almost periodic functions, then the sequence $\{n\alpha + \sum_j f_j(n)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$.*

Proof. As at the end of the proof of Proposition 1.2, we may select each f_j to be a trigonometric polynomial, and then, by considering individual terms $\cos(\lambda x)$, $\sin(\lambda x)$ in the f_j 's and by increasing ν we may also assume that each f_j is periodic (say of the form $c \cos(\lambda x)$ or $c \sin(\lambda x)$). Let β_j be the period of f_j , and set $\gamma_j = 1/\beta_j$. By replacing f_j by $g_j(x) = f(\beta_j x)$ it is sufficient to show that if each g_j is periodic with period 1, then the sequence $\{n\alpha + \sum_j g_j(n\gamma_j)\}_{n=1}^{\infty}$ is dense in $[0, 1) \pmod{1}$.

Let us choose a maximal subset $\{\gamma_\lambda\}_{\lambda \in \Lambda}$ of $\{\gamma_1, \dots, \gamma_\nu\}$ for which $\{1, \alpha\} \cup \{\gamma_\lambda\}_{\lambda \in \Lambda}$ is a rationally independent set, i.e., 0 can be written as a rational linear combination of these numbers only when all coefficients are 0. Thus, for every $j \notin \Lambda$ there is a non-trivial dependence of the form

$$c_{j,0}\alpha + \sum_{\lambda \in \Lambda} c_{j,\lambda}\gamma_\lambda + d_j\gamma_j = c_j,$$

where the c 's are integers and $d_j \neq 0$ is an integer. With some large integer L set $A = L \prod_{j \notin \Lambda} |d_j|$ (when $\Lambda = \{1, \dots, \nu\}$, then interpret the product as 1), and consider numbers n of the form $n = Am$, $m = 1, 2, \dots$. If $x \in (0, 1)$ is given, then,

by the rational independence of the set $\{1, \alpha\} \cup \{\gamma_\lambda\}_{\lambda \in \Lambda}$, there is a subsequence $\mathcal{M}$ of the integers such that for $m \rightarrow \infty$, $m \in \mathcal{M}$ we have

$$m\alpha \rightarrow x \pmod{1}, \quad m\gamma_\lambda \rightarrow \frac{1}{2} \pmod{1}, \quad \lambda \in \Lambda.$$

Now

$$n\alpha + \sum_{j=1}^{\nu} g_j(n\gamma_j) = n\alpha + \sum_{j \in \Lambda} g_j(n\gamma_j) + \sum_{j \notin \Lambda} g_j \left(\frac{c_j n - c_{j,0} \alpha n - \sum_{\lambda \in \Lambda} c_{j,\lambda} \gamma_\lambda n}{d_j} \right)$$

and this, by the 1-periodicity of the functions g_j , for $n = Am$ equals

$$A\{m\alpha\} + \sum_{j \in \Lambda} g_j(A\{m\gamma_j\}) + \sum_{j \notin \Lambda} g_j \left(-c_{j,0}(A/d_j)\{m\alpha\} - \sum_{\lambda \in \Lambda} c_{j,\lambda}(A/d_j)\{m\gamma_\lambda\} \right)$$

$\pmod{1}$, and this latter expression tends to

$$Ax + \sum_{j \in \Lambda} g_j(A(1/2)) + \sum_{j \notin \Lambda} g_j \left(-c_{j,0}(A/d_j)x - \sum_{\lambda \in \Lambda} c_{j,\lambda}(A/d_j)(1/2) \right) \quad (3.1)$$

as $m \rightarrow \infty$, $m \in \mathcal{M}$.

From this we can conclude that the $\pmod{1}$ taken sequence

$$\left\{ n\alpha + \sum_{j=1}^{\nu} g_j(n\gamma_j) \right\}_{n=1}^{\infty}$$

is dense in the $\pmod{1}$ taken range of the continuous function

$$h(x) = Ax + \sum_{j \in \Lambda} g_j(A(1/2)) + \sum_{j \notin \Lambda} g_j \left(-c_{j,0}(A/d_j)x - \sum_{\lambda \in \Lambda} c_{j,\lambda}(A/d_j)(1/2) \right)$$

defined on $(0, 1)$, and it is sufficient to show that this $\pmod{1}$ taken range is dense in $[0, 1)$.

The function h can also be considered on the interval $[0, 1]$ where it is a continuous function, and, if $|f_j| \leq M$, $1 \leq j \leq \nu$, then $h(0) \leq \nu M$, while $h(1) \geq A - \nu M$. Therefore, if $A > 2\nu M + 2$, then its range contains an interval $[B, B + 1]$ for some integer B , and hence if we take its range $\pmod{1}$, then we get the whole interval $[0, 1)$.

This completes the proof. $\square$

3.1. The polynomial case

As a generalization of what has been said before, we shall now examine the density of the sequence

$$\left\{ P_0(n) + \sum_{j=1}^{\nu} f_j(P_j(n)) \right\}_{n=1}^{\infty} \quad (3.2)$$

(mod 1), where P_j , $1 \leq j \leq \nu$, are polynomials. As before, we may assume that $P_j(0) = 0$ for all j .

PROPOSITION 3.2. *Let $f_1, \dots, f_{\nu}$ be almost periodic functions and let $P_0, P_1, \dots, P_{\nu}$ be polynomials without constant terms. If P_0 has at least one irrational coefficient, then the sequence (3.2) is dense in $[0, 1)$ (mod 1).*

COROLLARY 3.3. *With the assumptions of the preceding proposition*

$$\limsup_{n \rightarrow \infty} \left| \{P_0(n)\} - \left\{ \sum_{j=1}^{\nu} f_j(P_j(n)) \right\} \right| \geq \frac{1}{2}.$$

This corollary says that the fractional parts $\{P_0(n)\}$, $n = 1, 2, \dots$, cannot be imitated by sums of the fractional part of the values of almost periodic functions taken at polynomial places. Note also that for $\nu = 1$ and $f_1 \equiv 1/2$ the equality is attained, so more than what is claimed cannot be stated.

The corollary is immediate, for if the limsup in question was smaller than $1/2$, then the fractional part of $P_0(n) - \sum_{j=1}^{\nu} f_j(P_j(n))$, $n = 1, 2, \dots$, would not accumulate at $1/2$, so this sequence (mod 1) would not be dense in $[0, 1)$ contradicting Proposition 3.2.

Proof of Proposition 3.2. We follow the preceding proof, and exactly as there we may assume that each f_j is a continuous periodic function with period 1. Let us choose a maximal subset $\{P_{\lambda}\}_{\lambda \in \Lambda}$ of $\{P_1, \dots, P_{\nu}\}$ for which $\{P_0\} \cup \{P_{\lambda}\}_{\lambda \in \Lambda}$ is a $\mathbf{Q}$ -independent set. Thus, for every $j \notin \Lambda$ there is a non-trivial dependence of the form

$$c_{j,0}P_0 + \sum_{\lambda \in \Lambda} c_{j,\lambda}P_{\lambda} + d_jP_j = R_j,$$

where the c 's are integers, $d_j \neq 0$ is an integer and R_j is a polynomial with integer coefficients. With some large integer L set again $A = L \prod_{j \notin \Lambda} |d_j|$ and define

$$P_j^*(x) = \frac{1}{A}P_j(Ax), \quad R_j^*(x) = \frac{1}{A}R_j(Ax).$$

The R_j^* has integer coefficients, and we claim that $\{P_0^*\} \cup \{P_\lambda^*\}_{\lambda \in \Lambda}$ is again a $\mathbf{Q}$ -independent set. Indeed, if this was not the case, then we could find integers a_0, a_λ , not all zero, such that

$$Q(x) = a_0 P_0^*(x) + \sum_{\lambda \in \Lambda} a_\lambda P_\lambda^*(x) = \frac{a_0}{A} P_0(Ax) + \sum_{\lambda \in \Lambda} \frac{a_\lambda}{A} P_\lambda(Ax)$$

has integer coefficients. But then for D equal to the degree of Q , the coefficients of

$$a_0 A^{D-1} P_0(x) + \sum_{\lambda \in \Lambda} a_\lambda A^{D-1} P_\lambda(x) = A^D Q(x/A)$$

would have integer coefficients, which is impossible by the choice of Λ .

Consider numbers n of the form $n = Am$, $m = 1, 2, \dots$. If $x \in (0, 1)$ is given, then, by the $\mathbf{Q}$ -independence of the set $\{P_0^*\} \cup \{P_\lambda^*\}_{\lambda \in \Lambda}$ and by Weyl's theorem, there is a subsequence $\mathcal{M}$ of the integers such that for $m \rightarrow \infty$, $m \in \mathcal{M}$, we have

$$P_0^*(m) \rightarrow x \pmod{1}, \quad P_\lambda^*(m) \rightarrow \frac{1}{2} \pmod{1}, \quad \lambda \in \Lambda.$$

Now

$$\begin{aligned} P_0(n) + \sum_{j=1}^{\nu} f_j(P_j(n)) &= P_0(n) + \sum_{j \in \Lambda} f_j(P_j(n)) \\ &+ \sum_{j \notin \Lambda} f_j \left(\frac{R_j(n) - c_{j,0} P_0(n) - \sum_{\lambda \in \Lambda} c_{j,\lambda} P_\lambda(n)}{d_j} \right). \end{aligned}$$

Here for all j and $n = Am$ we have

$$P_j(n) = AP_j^*(m), \quad R_j(n) = AR_j^*(m),$$

and since A is divisible by d_j , it follows that each $R_j(n)/d_j$ has integer coefficients. Thus, by the 1-periodicity of the functions f_j , for $n = Am$ the sum $P_0(n) + \sum_{j=1}^{\nu} f_j(P_j(n))$ equals

$$\begin{aligned} A\{P_0^*(m)\} + \sum_{j \in \Lambda} f_j(A\{P_j^*(m)\}) \\ + \sum_{j \notin \Lambda} f_j \left(-c_{j,0}(A/d_j)\{P_0^*(m)\} - \sum_{\lambda \in \Lambda} c_{j,\lambda}(A/d_j)\{P_\lambda^*(m)\} \right) \end{aligned}$$

(mod 1), and this latter expression tends to

$$Ax + \sum_{j \in \Lambda} f_j(A(1/2)) + \sum_{j \notin \Lambda} f_j \left(-c_{j,0}(A/d_j)x - \sum_{\lambda \in \Lambda} c_{j,\lambda}(A/d_j)(1/2) \right),$$

as $m \rightarrow \infty$, $m \in \mathcal{M}$.

This is now an analogue of (3.1), and from here the proof is completed exactly as the preceding proof was completed from (3.1). $\square$

Proposition 3.2 can be written in a seemingly more general, but actually equivalent form.

PROPOSITION 3.4. *Let F be an almost periodic function of ν variables³ and let $P_0, P_1, \dots, P_\nu$ be polynomials without constant terms. If P_0 has at least one irrational coefficient, then the sequence*

$$\{P_0(n) + F(P_1(n), \dots, P_\nu(n))\}_{n=1}^\infty \quad (3.3)$$

is dense in $[0, 1) \pmod{1}$.

P r o o f. One can replace F by a generalized trigonometric polynomial of several variables, and then one can follow the preceding proof without much change.

An alternative is to use that if F is a generalized trigonometric polynomial of ν variables, then $F(P_1(n), \dots, P_\nu(n))$ is of the form

$$\sum_{j=1}^{\nu_1} f_j(R_j(n))$$

where each f_j is a $c \cdot \cos \lambda x$ or a $c \cdot \sin \lambda x$, and where the R_j 's are polynomials (just use trigonometric identities to convert products to sums). This reduces then the claim to Proposition 3.2. $\square$

3.2. The vector case

Let $\alpha_1, \dots, \alpha_d$ be real numbers. Recall Weyl's theorem from Section 2 according to which for the $(\text{mod } 1)$ density in $[0, 1)^d$ of the vectors $(n\alpha_1, \dots, n\alpha_d)$, $n = 1, 2, \dots$, it is necessary and sufficient that the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent.

PROPOSITION 3.5. *If $f_1, \dots, f_d$ are almost periodic functions and the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent, then the sequence*

$$(n\alpha_1 + f_1(n), \dots, n\alpha_d + f_d(n)), \quad n = 1, 2, \dots, \quad (3.4)$$

is dense in $[0, 1)^d \pmod{1}$.

³The set of these functions $F : \mathbf{R}^\nu \rightarrow \mathbf{R}$ is the closure in the uniform norm of the algebra generated by the monomials $\cos \lambda x_j, \sin \lambda x_j, \lambda \in \mathbf{R}, 1 \leq j \leq \nu$.

Proof. The argument is familiar by now, see, e.g., the proof of Proposition 3.1.

By replacing each f_j by a trigonometric polynomial close to it, it is enough to prove that if $f_{i,j}$, $1 \leq i \leq d$, $1 \leq j \leq \nu$, are continuous periodic functions then for any $\beta_{i,j}$ the sequence

$$\left(n\alpha_1 + \sum_{j=1}^{\nu} f_{1,j}(n\beta_{1,j}), \dots, n\alpha_d + \sum_{j=1}^{\nu} f_{d,j}(n\beta_{d,j}) \right), \quad n = 1, 2, \dots, \quad (3.5)$$

is dense in $[0, 1)^d \pmod{1}$. By applying dilation in $f_{i,j}$ and changing at the same time the corresponding $\beta_{i,j}$, we may assume that each $f_{i,j}$ is of period 1.

Select a maximal subset $\Lambda \subset \{(i, j) \mid 1 \leq i \leq d, 1 \leq j \leq \nu\}$ such that

$$\{1, \alpha_1, \dots, \alpha_d\} \cup \{\beta_\lambda \mid \lambda \in \Lambda\} \quad (3.6)$$

is a rationally independent set. Then every $\beta_{i,j}$ with $(i, j) \notin \Lambda$ can be expressed as a rational combination of the numbers in (3.6), say (with $\alpha_0 = 1$)

$$\beta_{i,j} = \sum_{\tau=0}^d c_\tau^{i,j} \alpha_\tau + \sum_{\lambda \in \Lambda} d_\lambda^{i,j} \beta_\lambda.$$

If q is the least common denominator of the coefficients in these representations, then for some large L set $A = Lq$ and consider the numbers $n = Am$, $m = 1, 2, \dots$, for which

$$\left(n\alpha_1 + \sum_{j=1}^{\nu} f_{1,j}(n\beta_{1,j}), \dots, n\alpha_d + \sum_{j=1}^{\nu} f_{d,j}(n\beta_{d,j}) \right)$$

is of the form

$$\left(Am\alpha_1 + \sum_{j=1}^{\nu} f_{1,j}(m(\dots)), \dots, Am\alpha_d + \sum_{j=1}^{\nu} f_{d,j}(m(\dots)) \right) \quad (3.7)$$

where $(\dots)$ represent integral combinations of the numbers in (3.6).

By Weyl's theorem, if $(x_1, \dots, x_d) \in (0, 1)^d$, then there is a subsequence $\mathcal{M}$ of the natural numbers such that, as $m \rightarrow \infty$, $m \in \mathcal{M}$, we have

$$\{m\alpha_j\} \rightarrow x_j, \quad 1 \leq j \leq d, \quad m\beta_\lambda \rightarrow \frac{1}{2}, \quad \lambda \in \Lambda,$$

and we obtain from (3.7) that the sequence (3.4) is dense $\pmod{1}$ in the $\pmod{1}$ taken range of the function

$$H(x_1, \dots, x_d) = \left(Ax_1 + \sum_{j=1}^{\nu} f_{1,j}(\dots), \dots, Ax_d + \sum_{j=1}^{\nu} f_{d,j}(\dots) \right) \quad (3.8)$$

for $(x_1, \dots, x_d) \in [0, 1]^d$, where now $\dots$ stand for some integral linear combinations of $x_1, \dots, x_d, 1/2$.

Let $|f_{i,j}| \leq M$ and $A > 2\nu M + 2$. By the following lemma (apply it to

$$h(x_1, \dots, x_d) = H(x_1/A, \dots, x_d/A)$$

and to the cube $Q = [0, A]^d$ the range of H over $[0, 1]^d$ contains a cube of side-length 1, hence the (mod 1) taken range of H contains $[0, 1]^d$, from which the claim in the proposition follows. $\square$

In the next lemma we set $\underline{x} = (x_1, \dots, x_d)$.

LEMMA 3.6. *If*

$$h(\underline{x}) = (x_1 + h_1(\underline{x}), \dots, x_d + h_d(\underline{x}))$$

is a continuous mapping of a cube $Q \subset \mathbf{R}^d$ of side-length A into $\mathbf{R}^d$, where $|h_i| \leq A/2 - 1$, then the range of h contains a (closed) cube of side-length 1.

Proof. This is an easy consequence of the Brouwer fixed point theorem. Indeed, we may assume that $Q = [-A/2, A/2]^d$, and we show that then the range contains the cube $[-1/2, 1/2]^d$.

Suppose to the contrary that this is not the case, and there is a point $\underline{y} \in [-1/2, 1/2]^d$ which is not in the range. For an $\underline{x} \in Q$ connect $h(\underline{x})$ with $\underline{y}$, and let $\Phi(\underline{x})$ be the intersection of the extension of this segment (beyond $\underline{y}$) with the boundary ∂Q of Q . This defines a continuous mapping $\Phi : Q \rightarrow \partial Q$. Note that if $\underline{x}$ belongs to the boundary of Q , say $x_1 = -A/2$, then, since $x_1 + h_1(\underline{x}) \leq -1$, in forming $\Phi(\underline{x})$ we are connecting the point $h(\underline{x})$ lying in the half-space $x_1 \leq -1$ with the point $\underline{y}$ lying in the half-space $x_1 \geq -1/2$. This implies (say by the convexity of the half-space $x_1 < -1/2$) that the first coordinate of $\Phi(\underline{x})$ is $\geq -1/2$, and hence $\Phi(\underline{x}) \neq \underline{x}$. Thus, Φ is a continuous mapping from Q into itself without a fixed point, which is impossible by the Brouwer fixed point theorem.

This contradiction proves the lemma. $\square$

3.3. The polynomial-vector case

The most general theorem in this paper on density is the following (cf. also Theorem 6.2 below).

THEOREM 3.7. *Let $F_1, \dots, F_d$ be almost periodic functions of ν variables, and let $P_{i,j}$ and $P_1, \dots, P_d$ be polynomials without constant terms. If $\{P_1, \dots, P_d\}$ are $\mathbf{Q}$ -independent, then the sequence of vectors*

$$\left\{ \left(P_1(n) + F_1(P_{1,1}(n), \dots, P_{1,\nu}(n)), \dots, P_d(n) + F_d(P_{d,1}(n), \dots, P_{d,\nu}(n)) \right) \right\}_{n=1}^{\infty} \quad (3.9)$$

is dense in $[0, 1)^d \pmod{1}$.

Recall that the $\mathbf{Q}$ -independence of $\{P_1, \dots, P_d\}$ is necessary for density when each F_i is identically 0.

Proof. All components of the proof had already been demonstrated in our previous discussions.

First of all, we may replace each F_i by a trigonometric polynomial of ν variables (see the end of the proof of Proposition 1.2), then using trigonometric identities, each $F_i(P_{i,1}(n), \dots, P_{i,\nu}(n))$ can be written in the form $\sum_{j=1}^{\mu} \varphi_i(S_{i,j})$ where each φ_i is a $c \cdot \cos \lambda t$ or a $c \cdot \sin \lambda t$ and $S_{i,j}$ are polynomials, and finally such an expression is of the form $\tilde{F}_i(S_{i,1}(n), \dots, S_{i,\mu}(n))$, where $\tilde{F}_i$ is a continuous function of μ variables which is periodic in each variable. Thus, we may assume from the start that F_i is periodic in each of its variable, and by scaling we may even assume that the periods are 1.

Select a maximal subset $\Lambda \subset \{(i, j) \mid 1 \leq i \leq d, 1 \leq j \leq \nu\}$ such that

$$\{P_1, \dots, P_d\} \cup \{P_\lambda \mid \lambda \in \Lambda\} \quad (3.10)$$

is a $\mathbf{Q}$ -independent set of polynomials. Then for every $P_{i,j}$ with $(i, j) \notin \Lambda$ there is a relation with the polynomials in (3.10) of the form

$$\sum_{\tau=1}^d c_\tau^{i,j} P_\tau + \sum_{\lambda \in \Lambda} c_\lambda^{i,j} P_\lambda + d_{i,j} P_{i,j} = R_{i,j},$$

where the c 's are integers, $d_{i,j} \neq 0$ is an integer and $R_{i,j}$ is a polynomial with integer coefficients. We consider again with some large integer L the number $A = L \prod_{(i,j) \notin \Lambda} |d_{i,j}|$ and define for a polynomial P the polynomial $P^*(x) = \frac{1}{A} P(Ax)$. The $*$ -transform of the system (3.10) is again $\mathbf{Q}$ -independent (see the proof of Proposition 3.2), so, by Weyl's theorem, if $(x_1, \dots, x_d) \in (0, 1)^d$ is given, then there is a subsequence $\mathcal{M}$ of the integers such that for $m \rightarrow \infty$, $m \in \mathcal{M}$ we have

$$P_j^*(m) \rightarrow x_j \pmod{1}, \quad j = 1, \dots, d, \quad P_\lambda^*(m) \rightarrow \frac{1}{2} \pmod{1}, \quad \lambda \in \Lambda.$$

By considering numbers of the form $n = Am$, $m = 1, 2, \dots$, and following the steps in the proof of Propositions 3.2 and 3.5 we can conclude that the closure of the $(\text{mod } 1)$ taken sequence (3.9) includes the $(\text{mod } 1)$ taken range of the function

$$H(x_1, \dots, x_d) = (Ax_1 + F_1(\dots), \dots, Ax_d + F_d(\dots))$$

for $(x_1, \dots, x_d) \in [0, 1]^d$, where, for each of the variables of each F_i , the $\dots$ stand for some integral linear combinations of $x_1, \dots, x_d, 1/2$. That this $(\text{mod } 1)$ taken range is $[0, 1]^d$ follows from Lemma 3.6 precisely as in the proof of Proposition 3.5. $\square$

4. Uniform distribution

In this section we shall examine the uniform distribution of the sequences that have been considered before. In the classical case Weyl's theorems implies that the conditions for density and uniform distribution are the same. We shall see that when a periodic perturbation is present the situation is different: density is always true (as we have seen in the preceding section), but this is not the case for uniform distribution.

4.1. The scalar case

Let us start with the simplest possible situation.

PROPOSITION 4.1. *Let α be irrational and let f be a β -periodic continuous function. Then the sequence $\{n\alpha + f(n)\}_{n=1}^{\infty}$ is uniformly distributed (mod 1) if β is rational or $1, \alpha, 1/\beta$ are rationally independent.*

Let us mention that Proposition 4.1 is precise in the sense that if β is irrational and $1, \alpha, 1/\beta$ are rationally dependent, then there is a β -periodic continuous function f such that the sequence $\{n\alpha + f(n)\}_{n=1}^{\infty}$ is not uniformly distributed (mod 1). See Proposition 5.1.

Proof. Set $\gamma = 1/\beta$ and $g(x) = f(\beta x)$. Then g is 1-periodic and $n\alpha + f(n) = \{n\alpha\} + g(\{n\gamma\})$ (mod 1).

Consider first the case when $\gamma = 1/\beta$ is rational, say $\gamma = p/q$. Then for $n = mq + s$, $0 \leq s < q$, we have

$$n\alpha + g(n\gamma) = \{m(q\alpha)\} + (g(s\gamma) + s\alpha) \pmod{1},$$

and since here the second term on the right is independent of m and $q\alpha$ is irrational, it follows from the uniform distribution of $\{m(q\alpha)\}_{m=1}^{\infty}$ (mod 1) that the sequence $\{(mq + s)\alpha + f((mq + s))\}_{m=1}^{\infty}$ is uniformly distributed (mod 1). Since this is true for all $0 \leq s < q$, the (mod 1) uniform distribution of $\{n\alpha + f(n)\}_{n=1}^{\infty}$ follows.

Let us now consider the case when $1, \alpha, 1/\beta$, i.e., $1, \alpha, \gamma$ are rationally independent. Then the vector sequence $\{(n\alpha, n\gamma)\}_{n=1}^{\infty}$ is uniformly distributed in $[0, 1)^2$ (mod 1).

Let $0 < a < b < 1$ be fixed, and for an $0 < \varepsilon < \min\{a, 1 - b\}$ let m be so large that the oscillation⁴ of g over any interval of length $< 1/m$ is smaller than ε .

⁴Here and in what follows, the oscillation of a function on a set is the difference of its supremum and infimum on that set

For $s = 0, 1, \dots, m-1$ and for an n consider the sets

$$K_{1,s} = \left\{ 1 \leq k \leq n \mid k\gamma \in \left[\frac{s}{m}, \frac{s+1}{m} \right], k\alpha \in \left[a - g\left(\frac{s}{m}\right) + \varepsilon, b - g\left(\frac{s}{m}\right) - \varepsilon \right] \right\},$$

and

$$K_{2,s} = \left\{ 1 \leq k \leq n \mid k\gamma \in \left[\frac{s}{m}, \frac{s+1}{m} \right], k\alpha \in \left[a - g\left(\frac{s}{m}\right) - \varepsilon, b - g\left(\frac{s}{m}\right) + \varepsilon \right] \right\},$$

where the relations are understood (mod 1). Since for

$$k\gamma \in \left[\frac{s}{m}, \frac{s+1}{m} \right] \pmod{1}$$

we have

$$g(k\gamma) \in \left[g\left(\frac{s}{m}\right) - \varepsilon, g\left(\frac{s}{m}\right) + \varepsilon \right]$$

by the choice of m , it follows that for $k \in K_{1,s}$ we have $k\alpha + g(k\gamma) \in [a, b] \pmod{1}$, and on the other hand, if $k\gamma \in [\frac{s}{m}, \frac{s+1}{m}]$ and $k\alpha + g(k\gamma) \in [a, b] \pmod{1}$, then $k \in K_{2,s}$. By the (mod 1) uniform distribution of the sequence of vectors $\{(k\alpha, k\gamma)\}_{k=1}^{\infty}$ we have

$$\#K_{1,s} = (1 + o(1))n \frac{b - a - 2\varepsilon}{m}, \quad \#K_{2,s} = (1 + o(1))n \frac{b - a + 2\varepsilon}{m},$$

hence it follows by summing these for all $s = 0, 1, \dots, m-1$ that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq k \leq n \mid k\alpha + g(k\gamma) \in [a, b] \pmod{1}\} \geq b - a - 2\varepsilon$$

and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \# \{1 \leq k \leq n \mid k\alpha + g(k\gamma) \in [a, b] \pmod{1}\} \leq b - a + 2\varepsilon.$$

Since $\varepsilon > 0$ and $0 < a < b < 1$ are arbitrary, the uniform distribution of $\{n\alpha + f(n)\}_{n=1}^{\infty} \pmod{1}$ follows. $\square$

We shall need the extension of Proposition 4.1 to more than one term. To this end, note that for an irrational α the condition “ β is rational or $1, \alpha, 1/\beta$ are rationally independent” can be phrased as “ α cannot be written as a rational linear combination of 1 and $1/\beta$ ”. It turns out that this is the right formulation in the general case.

PROPOSITION 4.2. *Let α be irrational and let $f_1, \dots, f_\nu$ be continuous periodic functions with periods $\beta_1, \dots, \beta_\nu$. If α cannot be written as a rational linear combination of the numbers $1, 1/\beta_1, \dots, 1/\beta_\nu$, then the sequence $\{n\alpha + f_1(n) + \dots + f_\nu(n)\}_{n=1}^{\infty}$ is uniformly distributed (mod 1).*

This is best possible in the sense that if α and β_j are given and α can be written as a rational linear combination of the numbers $1, 1/\beta_1, \dots, 1/\beta_\nu$, then there are β_j -periodic continuous functions f_j for $j = 1, \dots, \nu$, such that the sequence $n\alpha + f_1(n) + \dots + f_\nu(n)$, $n = 1, 2, \dots$, is not uniformly distributed (mod 1). See Proposition 5.1.

Proof. A large part of the proof has been already established in the preceding proof.

Set $\gamma_i = 1/\beta_i$ and $g_i(x) = f_i(\beta_i x)$. Then g_i are 1-periodic and

$$n\alpha + f_1(n) + \dots + f_\nu(n) = \{n\alpha + g_1(n\gamma_1) + \dots + g_\nu(n\gamma_\nu)\} \pmod{1}.$$

Select a maximal subset of $\gamma_1, \dots, \gamma_\nu$ that forms with 1 a rationally independent set. We may assume that $\{\gamma_1, \dots, \gamma_\mu\}$ is this maximal subset. Then, by the assumption of the theorem, $1, \alpha, \gamma_1, \dots, \gamma_\mu$ are rationally independent, furthermore every γ_j with $j > \mu$ is a rational combination of the form

$$\gamma_j = d_j + \sum_{i=1}^{\mu} c_{j,i} \gamma_i. \quad (4.1)$$

Let q be a common multiple of the denominators of these d_j . If k is of the form $k = ql + r$, $0 \leq r < q$, then

$$k\gamma_j = rd_j + \sum_{i=1}^{\mu} c_{j,i} k\gamma_i \pmod{1},$$

and it is sufficient to show the uniform distribution in question along every sequence $\mathcal{N}_r = \{ql + r\}_{l=1}^{\infty}$ where $0 \leq r < q$ is fixed.

For a large m partition $[0, 1]^\mu$ by the hyperplanes $x_i = t/m$, $t = 1, \dots, m-1$, $1 \leq i \leq \mu$, in $\mathbf{R}^\mu$ into m^μ little cubes I_s , $1 \leq s \leq m^\mu$. The image of such an I_s under the mapping

$$(x_1, \dots, x_\mu) \rightarrow rd_j + \sum_{i=1}^{\mu} c_{j,i} x_i \quad (4.2)$$

(cf. (4.1)) is an interval $I_{j,s}$ of length $\leq C/m$.

Now let $0 < a < b < 1$ be fixed, and for an $0 < \varepsilon < \min\{a, 1-b\}$ let m be so large that the oscillation of $g_1(x_1) + \dots + g_\mu(x_\mu)$ over any I_s , as well as the oscillation of each g_j , $j > \mu$, over any $I_{j,s}$ is smaller than $\varepsilon/2\nu$. Let, furthermore, $Q_s = (Q_{1,s}, \dots, Q_{\mu,s})$ be an arbitrary point from I_s , let $\tilde{Q}_{j,s}$ be the image of Q_s under the mapping (4.2), and set

$$T_s = \sum_{i=1}^{\mu} g_i(\gamma_i Q_{i,s}) + \sum_{j=\nu+1}^d g_j(\gamma_j \tilde{Q}_{j,s}).$$

For $s = 1, \dots, m^\mu$ consider the sets

$$K_{1,s} = \{1 \leq k \leq n, k \in \mathcal{N}_r \mid (k\gamma_1, \dots, k\gamma_\mu) \in I_s, k\alpha \in [a - T_s + \varepsilon, b - T_s - \varepsilon]\},$$

and

$$K_{2,s} = \{1 \leq k \leq n, k \in \mathcal{N}_r \mid (k\gamma_1, \dots, k\gamma_\mu) \in I_s, k\alpha \in [a - T_s - \varepsilon, b - T_s + \varepsilon]\},$$

where the relations are understood (mod 1). Exactly as in the preceding proof, by the choice of m , it follows that for $k \in K_{1,s}$ we have

$$k\alpha + g_1(\gamma_1 k) + \dots + g_\nu(\gamma_\nu k) \in [a, b] \pmod{1}, \quad (4.3)$$

and on the other hand, if $(k\gamma_1, \dots, k\gamma_\mu) \in I_s$ and (4.3) is true, then $k \in K_{2,s}$. By the (mod 1) uniform distribution of the sequence of vectors

$$\{(k\alpha, k\gamma_1, \dots, k\gamma_\mu)\}_{k=1, k \in \mathcal{N}_r}^\infty$$

(which follows from the rational independence of $(1, \alpha, \gamma_1, \dots, \gamma_\mu)$), we have

$$\#K_{1,s} = (1 + o(1))n \frac{b - a - 2\varepsilon}{q \cdot m^\mu}, \quad \#K_{2,s} = (1 + o(1))n \frac{b - a + 2\varepsilon}{q \cdot m^\mu}.$$

It follows by summing these for all $s = 1, \dots, m^\mu$ that

$$\liminf_{n \rightarrow \infty} \frac{q}{n} \# \{1 \leq k \leq n, k \in \mathcal{N}_r \mid k\alpha + g_1(\gamma_1 k) + \dots + g_\nu(\gamma_\nu k) \in [a, b] \pmod{1}\} \geq b - a - 2\varepsilon$$

and

$$\limsup_{n \rightarrow \infty} \frac{q}{n} \# \{1 \leq k \leq n, k \in \mathcal{N}_r \mid k\alpha + g_1(\gamma_1 k) + \dots + g_\nu(\gamma_\nu k) \in [a, b] \pmod{1}\} \leq b - a + 2\varepsilon.$$

Since $\varepsilon > 0$ and $0 < a < b < 1$ are arbitrary and since there are $n/q + O(1)$ numbers $k \in \mathcal{N}_r$ with $1 \leq k \leq n$, the uniform distribution of

$$\{k\alpha + g_1(\gamma_1 k) + \dots + g_\nu(\gamma_\nu k)\}_{k=1, k \in \mathcal{N}_r}^\infty$$

(mod 1) follows. $\square$

Next we state a somewhat more general form (but actually, since the number ν of terms in Proposition 4.2 is arbitrary, and since, by Weierstrass theorem, every periodic continuous function of several variables can be uniformly approximated by trigonometric polynomials of the same variables, one can show that this seemingly more general form is equivalent to Proposition 4.2).

PROPOSITION 4.3. *Let α be irrational and let F be a continuous function of the variables $x_1, \dots, x_\nu$ which is periodic in each variable with periods $\beta_1, \dots, \beta_\nu$, respectively. If α cannot be written as a rational linear combination of the numbers $1, 1/\beta_1, \dots, 1/\beta_\nu$, then the sequence $\{n\alpha + F(n, \dots, n)\}_{n=1}^\infty$ is uniformly distributed (mod 1).*

In particular, if

$$F(x_1, \dots, x_\nu) = f_1(x_1) + \dots + f_\nu(x_\nu),$$

where each f_j is a continuous periodic function of period β_j , then we obtain Proposition 4.2.

Proof. We follow the preceding proof.

Set

$$\gamma_i = 1/\beta_i \quad \text{and} \quad G(x_1, \dots, x_\nu) = F(\beta_1 x_1, \dots, \beta_\nu x_\nu).$$

Then G is 1-periodic in each variable and

$$n\alpha + F(n, \dots, n) = \{n\alpha + G(\{n\gamma_1\}, \dots, \{n\gamma_\nu\})\} \pmod{1}.$$

Select again a maximal subset of $\gamma_1, \dots, \gamma_\nu$ that forms with 1 a rationally independent set. We may assume that $\{\gamma_1, \dots, \gamma_\mu\}$ is this maximal subset. Then, by the assumption of the theorem, $1, \alpha, \gamma_1, \dots, \gamma_\mu$ are rationally independent, furthermore every γ_j with $j > \mu$ is a rational combination of the form (4.1). For a large m partition $[0, 1]^\mu$ by the hyperplanes $x_i = t/m$, $t = 1, \dots, m-1$ in $\mathbf{R}^\mu$ into m^μ little cubes I_s , $1 \leq s \leq m^\mu$, as before. The image of such an I_s under the mapping (4.2) (cf. (4.1)) is again an interval $I_{j,s}$ of length $\leq C/m$.

Now for fixed $0 < a < b < 1$ and for an $0 < \varepsilon < \min\{a, 1-b\}$ let m be so large that the oscillation of G over any set of diameter $\leq C\nu/m$ is smaller than $\varepsilon/2$. Let furthermore, $Q_s = (Q_{s,1}, \dots, Q_{s,\mu})$ be an arbitrary point from I_s , let $\tilde{Q}_{j,s}$ be the image of Q_s under the mapping (4.2), and set

$$T_s = G(\gamma_1 Q_{s,1}, \dots, \gamma_\mu Q_{s,\mu}, \gamma_{\mu+1} \tilde{Q}_{\mu+1,s}, \dots, \gamma_\nu \tilde{Q}_{\nu,s}).$$

With these modifications for the sets

$$K_{1,s} = \{1 \leq k \leq n, \ k \in \mathcal{N}_r \mid (k\gamma_1, \dots, k\gamma_\mu) \in I_s, \ k\alpha \in [a - T_s + \varepsilon, b - T_s - \varepsilon]\},$$

and

$$K_{2,s} = \{1 \leq k \leq n, \ k \in \mathcal{N}_r \mid (k\gamma_1, \dots, k\gamma_\mu) \in I_s, \ k\alpha \in [a - T_s - \varepsilon, b - T_s + \varepsilon]\},$$

(where the relations are understood $\pmod{1}$) we get again that for $k \in K_{1,s}$ we have $k\alpha + G(\gamma_1 k, \dots, \gamma_\nu k) \in [a, b] \pmod{1}$, and on the other hand, if $(k\gamma_1, \dots, k\gamma_\mu) \in I_s$ and $k\alpha + G(\gamma_1 k, \dots, \gamma_\nu k) \in [a, b] \pmod{1}$, then $k \in K_{2,s}$. By the $\pmod{1}$ uniform distribution of the sequence of vectors $\{(k\alpha, k\gamma_1, \dots, k\gamma_\mu)\}_{k=1, k \in \mathcal{N}_r}^\infty$ (which is a consequence of the rational independence of $(1, \alpha, \gamma_1, \dots, \gamma_\mu)$), we get again

$$\#K_{1,s} = (1 + o(1))n \frac{b - a - 2\varepsilon}{q \cdot m^\mu}, \quad \#K_{2,s} = (1 + o(1))n \frac{b - a + 2\varepsilon}{q \cdot m^\mu},$$

and from here the uniform distribution of

$$\{k\alpha + G(k\gamma_1, \dots, k\gamma_\nu)\}_{k=1, k \in \mathcal{N}_r}^\infty$$

(mod 1) for each $0 \leq r < q$ follows as before, and this proves Proposition 4.3. $\square$

4.2. The polynomial case

Now we extend Proposition 4.3 to the polynomial case. As before, we may assume without loss of generality that the polynomials we are dealing with are without constant term.

PROPOSITION 4.4. *Let $P_0, P_1, \dots, P_d$ be polynomials without constant terms and let F be a continuous function of variables $x_1, \dots, x_\nu$ which is periodic in each variable with periods $\beta_1, \dots, \beta_\nu$, respectively. If P_0 cannot be written as a rational linear combination of the polynomials $P_1/\beta_1, \dots, P_d/\beta_d$ and of a polynomial with rational coefficients, then the sequence*

$$\{P_0(n) + F(P_1(n), \dots, P_d(n))\}_{n=1}^\infty \quad (4.4)$$

is uniformly distributed (mod 1).

Note that Proposition 4.3 is the special case of this when $P_0(x) = \alpha x$ and $P_j(x) = x$ for all other j .

In particular, if

$$F(x_1, \dots, x_\nu) = f_1(x_1) + \dots + f_\nu(x_\nu),$$

where each f_j is a continuous periodic function of period β_j , then under the conditions of the theorem the (mod 1) uniform distribution of the sequence

$$\{P_0(n) + f_1(P_1(n)) + \dots + f_\nu(P_\nu(n))\}_{n=1}^\infty$$

follows. In Proposition 5.2 we shall show that this is best possible in the sense that if $P_0, P_1, \dots, P_\nu$ are given and P_0 can be written as a rational linear combination of the polynomials $P_1/\beta_1, \dots, P_\nu/\beta_\nu$ and of a polynomial with rational coefficients, then there are continuous functions f_j of period β_j , $1 \leq j \leq \nu$, such that the sequence

$$P_0(n) + f_1(P_1(n)) + \dots + f_d(P_d(n)), \quad n = 1, 2, \dots,$$

is not uniformly distributed (mod 1).

P r o o f. The proof is almost identical to that of Proposition 4.2. Indeed, as there set $\gamma_i = 1/\beta_i$ and

$$G(x_1, \dots, x_\nu) = F(\beta_1 x_1, \dots, \beta_\nu x_\nu).$$

Then $F(P_1(n), \dots, P_\nu(n)) = G(\gamma_1 P_1(n), \dots, \gamma_\nu P_\nu(n))$. Select a maximal subset of $\gamma_1 P_1, \dots, \gamma_\nu P_\nu$ for which no non-trivial linear combination with integer coefficients produces a polynomial with integer coefficients. We may assume that $\gamma_1 P_1, \dots, \gamma_\mu P_\mu$ is this maximal subset. Then, by the assumption of the theorem,

$P_0, \gamma_1 P_1, \dots, \gamma_\mu P_\mu$ is $\mathbf{Q}$ -independent (see Definition 2.1), but every $\gamma_j P_j$ with $j > \nu$ is a combination of the form

$$\gamma_j P_j(x) = R_j(x) + \sum_{i=1}^{\mu} c_{j,i} \gamma_i P_i(x), \quad (4.5)$$

where the $c_{j,i}$ are rationals and R_j has rational coefficients. Let q be a common multiple of the denominators of all the coefficients in all R_j , $\mu < j \leq \nu$. It is sufficient to verify the uniform distribution of each subsequence of (4.4) for which the indices n belong to $\mathcal{N}_r = \{ql + r\}_{l=1}^{\infty}$ with a fixed $0 \leq r < q$. If $k = ql + r$ is such an index, then

$$\gamma_j P_j(k) = R_j(r) + \sum_{i=1}^{\mu} c_{j,i} \gamma_i P_i(k) \pmod{1},$$

and on the right $R_j(r)$ is independent of $l = 1, 2, \dots$. This $R_j(r)$ plays the role of rd_j in (4.2). Since $P_0, \gamma_1 P_1, \dots, \gamma_\mu P_\mu$ are $\mathbf{Q}$ -independent, it follows that the polynomials $P_0(qx + r), \gamma_1 P_1(qx + r), \dots, \gamma_\mu P_\mu(qx + r)$ are also $\mathbf{Q}$ -independent (note that if a polynomial R has rational coefficients then so does $R((y-r)/q)$). Hence Weyl's theorem ensures that the sequence

$$(P_0(k), \gamma_1 P_1(k), \dots, \gamma_\mu P_\mu(k)), \quad k = ql + r, \quad l = 1, 2, \dots,$$

of vectors is uniformly distributed $\pmod{1}$ on $[0, 1)^{\nu+1}$. This is an analogue of the uniform distribution of the sequence $\{(k\alpha, n\gamma_1, \dots, k\gamma_\mu)\}_{k=1, k \in \mathcal{N}_r}^{\infty}$ in the preceding proof (while (4.5) is an analogue of (4.1)), and by replacing the latter by the previous one, that proof goes through without much change. $\square$

4.3. The vector case

Now we shall consider the vector case, namely the uniform distribution of vector sequences of the form

$$\{(n\alpha_1 + f_1(n), \dots, n\alpha_d + f_d(n))\}_{n=1}^{\infty} \quad (4.6)$$

$\pmod{1}$, where the f_i are continuous periodic functions. To this end we introduce the following definition.

DEFINITION 4.5. We say that a system $(\alpha_1, \beta_1), \dots, (\alpha_d, \beta_d)$ of number pairs is totally $\mathbf{Q}$ -independent, if for any non-empty subset $\Lambda \subset \{1, \dots, d\}$ it is true, that if $a_\lambda, \lambda \in \Lambda$, are nonzero integers, then the number $\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda$ cannot be written as a rational linear combination of the numbers $1, \beta_\lambda, \lambda \in \Lambda$.

As an example consider the pairs $(\sqrt{2}, \sqrt{3}), (\sqrt{3}, \pi), (\pi^2, \sqrt{2})$. This is easily seen to be totally $\mathbf{Q}$ -independent, though $\alpha_1 + \alpha_2 = \sqrt{2} + \sqrt{3} = \beta_1 + \beta_3$ (note that total $\mathbf{Q}$ -independence requires that $\alpha_1 + \alpha_2 = \sqrt{2} + \sqrt{3}$ should not be a rational linear combination of $\beta_1 = \sqrt{2}$ and $\beta_2 = \pi$ — we cannot use in this combination β_3).

PROPOSITION 4.6. *Let $f_1, \dots, f_d$ be continuous periodic functions with periods $\beta_1, \dots, \beta_d$. If $(\alpha_1, 1/\beta_1), \dots, (\alpha_d, 1/\beta_d)$ are totally $\mathbf{Q}$ -independent, then the sequence (4.6) is uniformly distributed on $[0, 1)^d \pmod{1}$.*

In Proposition 5.3 we shall show that one cannot do more than Proposition 4.6, for uniform distribution is not true (for some functions f_j of period β_j) if $(\alpha_1, 1/\beta_1), \dots, (\alpha_d, 1/\beta_d)$ are not totally $\mathbf{Q}$ -independent.

Proof. By Weyl's criterion, a sequence

$$(v_{1,n}, \dots, v_{d,n}), \quad n = 1, 2, \dots, \quad (4.7)$$

of vectors is uniformly distributed on $[0, 1)^d \pmod{1}$ precisely if for all integers $a_1, \dots, a_d$, $\sum a_j^2 > 0$, the sequence of numbers

$$a_1 v_{1,n} + \dots + a_d v_{d,n}, \quad n = 1, 2, \dots, \quad (4.8)$$

is uniformly distributed on $[0, 1) \pmod{1}$. Apply this to $v_{j,n} = \alpha_j n + f_j(n)$. If in the combination the non-zero a_j are a_λ , $\lambda \in \Lambda$, then we need to prove the $(\text{mod } 1)$ uniform distribution on $[0, 1)$ of the sequence

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda \right) n + \sum_{\lambda \in \Lambda} a_\lambda f_\lambda(n), \quad n = 1, 2, \dots,$$

which follows from Proposition 4.2 because, by total $\mathbf{Q}$ -independence, $\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda$ cannot be written as a rational linear combination of the numbers $1, 1/\beta_\lambda$, $\lambda \in \Lambda$. $\square$

We can state a more general form of the preceding proposition. To this end we introduce the following definition.

DEFINITION 4.7. We say that a system

$$(\alpha_1, \beta_{1,1}, \dots, \beta_{1,\nu}), \dots, (\alpha_d, \beta_{d,1}, \dots, \beta_{d,\nu})$$

of $(\nu+1)$ -tuples of numbers is totally $\mathbf{Q}$ -independent, if for any non-empty subset $\Lambda \subset \{1, \dots, d\}$ it is true, that if a_λ , $\lambda \in \Lambda$, are *nonzero* integers, then the number $\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda$ cannot be written as a rational linear combination of the numbers $1, \beta_{\lambda,j}$, $\lambda \in \Lambda, 1 \leq j \leq \nu$.

PROPOSITION 4.8. *Let $F_1, \dots, F_d$ be continuous periodic functions of ν variables such that F_i is periodic in the j th variable with period $\beta_{i,j}$. If*

$$(\alpha_1, 1/\beta_{1,1}, \dots, 1/\beta_{1,\nu}), \dots, (\alpha_d, 1/\beta_{d,1}, \dots, 1/\beta_{d,\nu})$$

are totally $\mathbf{Q}$ -independent, then the sequence

$$\{(n\alpha_1 + F_1(n, \dots, n), \dots, n\alpha_d + F_d(n, \dots, n))\}_{n=1}^{\infty} \quad (4.9)$$

of vectors is uniformly distributed on $[0, 1)^d \pmod{1}$.

Proof. Just follow the proof of Proposition 4.6. In this case we apply Weyl's criterion to a linear combination $\sum_j a_j v_{j,n}$ with $v_{j,n} = \alpha_j n + F_j(n, \dots, n)$. If in the combination the non-zero a_j are a_λ , $\lambda \in \Lambda$, then we need to prove the $\pmod{1}$ uniform distribution on $[0, 1)$ of the sequence

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda \right) n + \sum_{\lambda \in \Lambda} a_\lambda F_\lambda(n, \dots, n), \quad n = 1, 2, \dots,$$

which follows from Proposition 4.3 because, by total $\mathbf{Q}$ -independence, $\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda$ cannot be written as a rational linear combination of the numbers

$$1, 1/\beta_{\lambda,j}, \quad \lambda \in \Lambda, \quad 1 \leq j \leq \nu.$$

Indeed, if, say, $\Lambda = \{1, \dots, \kappa\}$ (which we may assume), then one should replace ν in Proposition 4.3 by $\kappa\nu$ and apply Proposition 4.3 to the function

$$F(x_1, \dots, x_{\kappa\nu}) = F_1(x_1, \dots, x_\nu) + F_2(x_{\nu+1}, \dots, x_{2\nu}) + \dots + F_\kappa(x_{(\kappa-1)\nu+1}, \dots, x_{\kappa\nu})$$

which is $\beta_{i,j}$ -periodic in the variable $x_{(i-1)\nu+j}$. $\square$

The polynomial version of Proposition 4.6 is our next result.

DEFINITION 4.9. We say that a system

$$(P_1, P_{1,1}, \dots, P_{1,\nu}), \dots, (P_d, P_{d,1}, \dots, P_{d,\nu})$$

of polynomial $(\nu + 1)$ -tuples is totally $\mathbf{Q}$ -independent, if for any non-empty subset $\Lambda \subset \{1, \dots, d\}$ it is true, that if a_λ , $\lambda \in \Lambda$ are *nonzero* integers, then the polynomial $\sum_{\lambda \in \Lambda} a_\lambda P_\lambda$ cannot be written as a linear combination with rational coefficients of the polynomials $P_{\lambda,j}$, $\lambda \in \Lambda$, $1 \leq j \leq \nu$, and of a polynomial with integer coefficients.

The most general result on distribution in this paper is the following.

THEOREM 4.10. *Let $F_1, \dots, F_d$ be continuous functions of ν variables such that each F_i is periodic in each of its variables with periods $\beta_{i,1}, \dots, \beta_{i,\nu}$, respectively. Let furthermore, $P_1, \dots, P_d$ and $P_{i,j}$, $1 \leq i \leq d$, $1 \leq j \leq \nu$, be polynomials without constant terms. If the system*

$$(P_1, P_{1,1}/\beta_{1,1}, \dots, P_{1,\nu}/\beta_{1,\nu}), \dots, (P_d, P_{d,1}/\beta_{d,1}, \dots, P_{d,\nu}/\beta_{d,\nu})$$

is totally $\mathbf{Q}$ -independent, then the sequence

$$\left(P_1(n) + F_1(P_{1,1}(n), \dots, P_{1,\nu}(n)), \dots, P_d(n) + F_d(P_{d,1}(n), \dots, P_{d,\nu}(n)) \right), \quad (4.10)$$

$n = 1, 2, \dots$, of vectors is uniformly distributed on $[0, 1)^d \pmod{1}$.

See Proposition 5.4 for the sharpness of this theorem.

Proof. The claim follows from Proposition 4.4 exactly as Proposition 4.6 resp., 4.8 followed from Proposition 4.2 resp., Proposition 4.6, just use again that uniform distribution of a sequence (4.7) is equivalent to the uniform distribution of all the sequences (4.8), and apply Proposition 4.4. $\square$

5. Exactness of the conditions for uniform distribution

In this section we show that the conditions for uniform distribution set forth in the preceding section are exact. Let us start by showing that the conditions in Proposition 4.1 and Proposition 4.2 cannot be relaxed.

PROPOSITION 5.1. *If α and $\beta_1, \dots, \beta_\nu$ are given and α can be written as a rational linear combination of the numbers $1, 1/\beta_1, \dots, 1/\beta_\nu$, then there are β_j -periodic continuous functions f_j for $j = 1, \dots, \nu$, such that the sequence $n\alpha + f_1(n) + \dots + f_\nu(n)$, $n = 1, 2, \dots$, is not uniformly distributed (mod 1).*

This proposition does not tell us if, in the case when uniform distribution does not happen, the sequence has a distribution or not. We shall briefly discuss that question in the next section.

Proof of Proposition 5.1. First of all, we may assume that α is irrational, for otherwise the claim is trivial by setting $f_j \equiv 0$.

Let $\gamma_j = 1/\beta_j$. By the assumption there are integers $q > 0$ and $p_0, p_1, \dots, p_\nu$ such that

$$\alpha + \sum_{j=1}^{\nu} \frac{p_j}{q} \gamma_j = \frac{p_0}{q}. \quad (5.1)$$

Multiply this equality by n and write each number on the left as its integral and fractional part to obtain

$$[n\alpha] + \{n\alpha\} + \sum_{j=1}^{\nu} \frac{p_j}{q} ([n\gamma_j] + \{n\gamma_j\}) = \frac{p_0}{q},$$

from which we can see that

$$\{n\alpha\} + \sum_{j=1}^{\nu} \frac{p_j}{q} \{n\gamma_j\} = \frac{s}{q},$$

where s is an integer. Since on the left we have a linear combination of numbers lying in $[0, 1)$, actually

$$|s| < |q| + |p_1| + \dots + |p_\nu| =: M.$$

Therefore, for every N there is an integer $t \in [-M, M]$ such that for at least $N/(2M+1)$ of the numbers $1 \leq n \leq N$ we have

$$\{n\alpha\} + \sum_{j=1}^{\nu} \frac{p_j}{q} \{n\gamma_j\} = \frac{t}{q}, \quad (5.2)$$

and we may assume that t is the same for infinitely many N .

Let $\varepsilon > 0$ be a small number. Since α is irrational, by the uniform distribution of the sequence $\{n\alpha\}$, $n = 1, 2, \dots$, for large N at least $(1-2\varepsilon)N$ of the numbers $\{n\alpha\}$, $n = 1, 2, \dots, N$, belong to $[0, 1-\varepsilon]$. In a similar vein, if for some j the number γ_j is irrational, then for large N at least $(1-2\varepsilon)N$ of the numbers $\{n\gamma_j\}$, $n = 1, 2, \dots, N$, belong to $[0, 1-\varepsilon]$. On the other hand, if γ_j is rational, then for small $\varepsilon > 0$ all the numbers $\{n\gamma_j\}$, $n = 1, 2, \dots, N$, belong to $[0, 1-\varepsilon]$. Thus, for at least $(1-2(\nu+1)\varepsilon)N$ of the numbers $1 \leq n \leq N$ these relations hold simultaneously. Choose N sufficiently large so that all these are satisfied. If, in addition, $2(\nu+1)\varepsilon < 1/2(2M+1)$, then out of these $\geq (1-2(\nu+1)\varepsilon)N$ numbers $1 \leq n \leq N$ at least $N/2(2M+1)$ also satisfy (5.2) (recall that (5.2) was true for at least $N/(2M+1)$ of the $1 \leq n \leq N$).

Let now g_j , $j = 1, \dots, \nu$, be 1-periodic continuous functions such that

$$g_j(x) = \frac{p_j}{q}x \quad \text{for } x \in [0, 1-\varepsilon].$$

In view of (5.2), for at least $N/2(2M+1)$ of the numbers $1 \leq n \leq N$ we have

$$n\alpha + \sum_{i=1}^d g_i(n\gamma_i) = \frac{t}{q} \pmod{1},$$

and since this is true for infinitely many N , it follows that the numbers on the left for $n = 1, 2, \dots$ are not uniformly distributed $\pmod{1}$ (the uniform distribution does not have point masses).

We can complete the proof by setting $f_j(x) = g_j(\gamma_j x) = g_j(x/\beta_j)$ □

Next we show that the condition given in Proposition 4.4 is sharp.

PROPOSITION 5.2. *If the numbers $\beta_1, \dots, \beta_\nu$ and the polynomials $P_0, P_1, \dots, P_\nu$ are given and P_0 can be written as a rational linear combination of the polynomials $P_1/\beta_1, \dots, P_\nu/\beta_\nu$ and of a polynomial with rational coefficients, then for $j = 1, \dots, \nu$ there are periodic continuous functions f_j of period β_j such that sequence*

$$P_0(n) + f_1(P_1(n)) + \dots + f_\nu(P_\nu(n)), \quad n = 1, 2, \dots,$$

is not uniformly distributed $\pmod{1}$.

P r o o f. The proof is almost identical to that of Proposition 5.1. We may assume that P_0 has an irrational coefficient (otherwise just set $f_j \equiv 0$).

Now start with a relation

$$P_0(x) + \sum_{j=1}^d \frac{p_j}{q} P_j(x) \gamma_j = \frac{p_0}{q} R_j(x), \quad (5.3)$$

where $\gamma_j = 1/\beta_j$ and R_j is a polynomial with integer coefficients. This is an analogue of (5.1), and from here proceed as we reasoned from (5.1). We only need to mention that since P_0 has at least one irrational coefficient, by Weyl's theorem, for large N , at least $(1-2\varepsilon)N$ of the numbers $\{P_0(n)\}$, $n = 1, 2, \dots, N$, belong to $[0, 1 - \varepsilon]$, and similarly, if for some j the polynomial $P_j \gamma_j$ has an irrational coefficient, then for large N at least $(1-2\varepsilon)N$ of the numbers $\{P_j(n) \gamma_j\}$, $n = 1, 2, \dots, N$, belong to $[0, 1 - \varepsilon]$. On the other hand, if all coefficients of $P_j \gamma_j$ are rational, then for sufficiently small $\varepsilon > 0$ all the numbers $\{P_j(n) \gamma_j\}$, $n = 1, 2, \dots, N$, belong to $[0, 1 - \varepsilon]$. $\square$

The conditions set forth in the vector case in Proposition 4.6 are also best possible:

PROPOSITION 5.3. *Let $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_d$ be given numbers. If the pairs $(\alpha_1, 1/\beta_1), \dots, (\alpha_d, 1/\beta_d)$ are not totally $\mathbf{Q}$ -independent, then there are β_j -periodic continuous functions f_j such that the sequence (4.6) is not uniformly distributed (mod 1) on $[0, 1)^d$.*

P r o o f. We may assume that $1, \alpha_1, \dots, \alpha_d$ are rationally independent (if this is not the case, then for $f_j \equiv 0$ the sequence (4.6) is not uniformly distributed by Weyl's theorem).

By assumption there is a non-empty set $\Lambda \subset \{1, \dots, d\}$ and non-zero integers a_λ , $\lambda \in \Lambda$, such that

$$\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda + \sum_{\lambda \in \Lambda} \frac{p_\lambda}{q} \gamma_\lambda = \frac{p_0}{q} \quad (5.4)$$

where $\gamma_\lambda = 1/\beta_\lambda$ and p_λ, q, p_0 are integers. This is an analogue of (5.2), and if we follow the reasoning after (5.2) (note that we can, since $\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda$ is irrational by the rational independence of $1, \alpha_1, \dots, \alpha_d$), then we obtain that there are continuous functions h_λ of period β_λ such that the sequence

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda \right) n + \sum_{\lambda \in \Lambda} h_\lambda(n), \quad n = 1, 2, \dots,$$

is not (mod 1) uniformly distributed.

If we set $f_\lambda = h_\lambda/a_\lambda$, then it follows that the numerical sequence

$$\left(\sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda\right)n + \sum_{\lambda \in \Lambda} a_\lambda f_\lambda(n), \quad n = 1, 2, \dots,$$

is not $(\bmod 1)$ uniformly distributed on $[0, 1)$. But then, by Weyl's theorem, the vector sequence (4.6) (where, say, we set $f_j \equiv 0$ if $j \notin \Lambda$) is not uniformly distributed $(\bmod 1)$ in $[0, 1)^d$, either. $\square$

Finally, we show that Theorem 4.10 is sharp. For simplicity we shall formulate the sharpness only for the case when the functions F_i are of a single variable.

THEOREM 5.4. *Let $\beta_1, \dots, \beta_d$ be given numbers and let $Q_1, \dots, Q_d$ and $P_1, \dots, P_d$ be given polynomials without constant terms. If the system of polynomial pairs $(Q_1, P_1/\beta_1), \dots, (Q_d, P_d/\beta_d)$ is not totally $\mathbf{Q}$ -independent, then there are continuous periodic functions f_j of period β_j , $1 \leq j \leq d$, such that the sequence*

$$\left\{ \left(Q_1(n) + f_1(P_1(n)), \dots, Q_d(n) + f_d(P_d(n)) \right) \right\}_{n=1}^{\infty} \quad (5.5)$$

is not uniformly distributed $(\bmod 1)$ on $[0, 1)^d$.

Proof. We may assume that $Q_1, \dots, Q_d$ are $\mathbf{Q}$ -independent (otherwise, set $f_j \equiv 0$ and apply Weyl's theorem).

By assumption there is a non-empty set $\Lambda \subset \{1, \dots, d\}$ and non-zero integers a_λ , $\lambda \in \Lambda$, such that with $\gamma_\lambda = 1/\beta_\lambda$ we have

$$\sum_{\lambda \in \Lambda} a_\lambda Q_\lambda(x) + \sum_{\lambda \in \Lambda} \frac{p_\lambda}{q} P_\lambda(x) \gamma_\lambda = \frac{p_0}{q} R(x),$$

where p_0, p_λ, q are integers and the polynomial R has integer coefficients. This is now an analogue of (5.4) and (5.3), and from here the reasoning is the same that was given after (5.4) (taking into account the necessary modifications in the polynomial case as were given after (5.3) and taking also into account that $\sum a_\lambda Q_\lambda$ has an irrational coefficient since $Q_1, \dots, Q_d$ are $\mathbf{Q}$ -independent) by Proposition 5.2 there are continuous functions h_λ of period β_λ such that the sequence

$$\left(\sum_{\lambda \in \Lambda} a_\lambda Q_\lambda(n) \right) + \sum_{\lambda \in \Lambda} h_\lambda(P_\lambda(n)), \quad n = 1, 2, \dots,$$

is not $(\bmod 1)$ uniformly distributed. Now set $f_\lambda = h_\lambda/a_\lambda$, and complete the proof as in Proposition 5.3. $\square$

6. Density and distribution on the torus

In this section we change somewhat our perspective. So far we have considered modulo 1 values as fractional parts on the interval $[0, 1)$. But we can consider them also on the torus $\mathbf{R}/\mathbf{Z}$, i.e., when we identify the points 0 and 1 in $[0, 1]$. All our results so far on the (mod 1) density and uniform distribution hold also on the torus without any change. Until now we have been considering sequences (or vector sequences) of the form

$$P_0(0) + F(P_1(n), \dots, P_\nu(n)), \quad n = 1, 2, \dots, \quad (6.1)$$

where, in some cases, F was allowed to be almost periodic. If we only restrict our attention only to periodic F , then it is natural to work with mappings $G : (\mathbf{R}/\mathbf{Z})^\nu \rightarrow \mathbf{R}/\mathbf{Z}$ (or with $G : (\mathbf{R}/\mathbf{Z})^\nu \rightarrow (\mathbf{R}/\mathbf{Z})^d$ in the vector case) and consider sequences

$$G(P_1(n), \dots, P_\nu(n)), \quad n = 1, 2, \dots \quad (6.2)$$

on $\mathbf{R}/\mathbf{Z}$ (or on $(\mathbf{R}/\mathbf{Z})^d$ in the vector case), which leads to a somewhat more general setting. Indeed, if the F in (6.1) is periodic in each variable with periods $\beta_1, \dots, \beta_\nu$, respectively, then by setting

$$G(x_0, x_1, \dots, x_\nu) = x_0 + F(\beta_1 x_1, \dots, \beta_\nu x_\nu), \quad (x_0, x_1, \dots, x_\nu) \in (\mathbf{R}/\mathbf{Z})^{\nu+1}, \quad (6.3)$$

and

$$P_j^*(x) = P_j(x)/\beta_j,$$

we have

$$P_0(0) + F(P_1(n), \dots, P_\nu(n)) = G(P_0(n), P_1^*(n), \dots, P_\nu^*(n)) \pmod{1},$$

so (6.2) include, indeed, (mod 1) sequences of the form (6.1).

Any function $G : (\mathbf{R}/\mathbf{Z})^\nu \rightarrow \mathbf{R}/\mathbf{Z}$ can also be considered as a function $G : \mathbf{R}^\nu \rightarrow \mathbf{R}/\mathbf{Z}$ by stipulating $G(x_1, \dots, x_\nu) := G(\{x_1\}, \dots, \{x_\nu\})$, and this is indeed how we interpreted the values in (6.2). The torus $\mathbf{R}/\mathbf{Z}$ can also be identified with the unit circle under the mapping $x \rightarrow e^{2\pi i x}$. This allows us to speak of the rotation (or winding) number w of a continuous mapping $g : \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Z}$ which is defined as the total change of the argument in $e^{2\pi i g(x)}$ as x runs through the interval $[0, 1]$. Intuitively, w tells us how many times $e^{2\pi i g(x)}$ circles the origin as $e^{2\pi i x}$ makes one full circle in the positive (counterclockwise) direction. For example, if m is an integer, then the mapping $x \rightarrow mx$ has rotation number m , and all $g : \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Z}$ with rotation number m is homotope with the mapping $x \rightarrow mx$.

If $G := (\mathbf{R}/\mathbf{Z})^\nu \rightarrow \mathbf{R}/\mathbf{Z}$ is continuous and we fix $x_2, \dots, x_\nu$, then $g(x_1) = G(x_1, x_2, \dots, x_\nu)$ is a mapping from $\mathbf{R}/\mathbf{Z}$ into itself, and let w_1 be its rotation number. Since this is an integer which changes continuously as we change

$x_2, \dots, x_\nu$ continuously, it follows that w_1 is independent of how $x_2, \dots, x_\nu$ are fixed, and we call w_1 the rotation number of G with respect to x_1 . The rotation number w_j with respect to the variable x_j , $2 \leq j \leq \nu$, is defined similarly. For example, the mapping G in (6.3) has rotation number $w_0 = 1$ with respect to the variable x_0 and (because of the periodicity of the function F) rotation number 0 ($w_j = 0$) with respect to all other variables x_j , $1 \leq j \leq \nu$.

6.1. Density on the torus

With the just given definitions we are now ready to extend Proposition 3.4 (at least for periodic functions).

PROPOSITION 6.1. *Let $G := (\mathbf{R}/\mathbf{Z})^\nu \rightarrow \mathbf{R}/\mathbf{Z}$ be a continuous function with rotation numbers $w_1, \dots, w_\nu$, respectively, and let $P_1, \dots, P_\nu$ be arbitrary polynomials without constant terms. If $w_1 P_1 + \dots + w_\nu P_\nu$ has at least one irrational coefficient, then the sequence*

$$\left\{ G(P_1(n), \dots, P_\nu(n)) \right\}_{n=1}^{\infty} \quad (6.4)$$

is dense in $\mathbf{R}/\mathbf{Z}$.

Note that this is exact in the following sense: if all coefficients of $w_1 P_1 + \dots + w_\nu P_\nu$ are rational, then for

$$G(x_1, \dots, x_\nu) = w_1 x_1 + \dots + w_\nu x_\nu$$

(which clearly has rotation numbers $w_1, \dots, w_\nu$), the sequence in (6.4) is not dense in $\mathbf{R}/\mathbf{Z}$ (for it has only finitely many different terms).

Proof of Proposition 6.1. The claim easily follows from Proposition 3.4. In fact, let

$$F(x_1, \dots, x_\nu) = G(x_1, \dots, x_\nu) - (w_1 x_1 + \dots + w_\nu x_\nu).$$

Since all rotation numbers of this F are zero, F (considered as a function from $\mathbf{R}^\nu$ to $\mathbf{R}$) is a 1-periodic function in each of its variable. Furthermore, if we set

$$P_0 := w_1 P_1 + \dots + w_\nu P_\nu,$$

then for all n we have

$$G(P_1(n), \dots, P_\nu(n)) = P_0(n) + F(P_1(n), \dots, P_\nu(n)),$$

hence the claim in the proposition is a consequence of Proposition 3.4. □

In a similar vein, one can obtain the vector case.

THEOREM 6.2. *Let $G_i(x_{i,1}, \dots, x_{i,\nu})$, $1 \leq i \leq d$, be continuous functions of ν variables on $\mathbf{R}/\mathbf{Z}$, and let $P_{i,j}$, $1 \leq j \leq \nu$, $1 \leq i \leq d$, be polynomials without constant terms. If the rotation number of G_i with respect to the variable $x_{i,j}$ is $w_{i,j}$ and if the polynomials*

$$P_i = w_{i,1}P_{i,1} + \dots + w_{i,\nu}P_{i,\nu}, \quad i = 1, \dots, d, \quad (6.5)$$

are $\mathbf{Q}$ -independent, then the sequence of vectors

$$\left(G_i(P_{i,1}(n), \dots, P_{i,\nu}(n)) \right)_{i=1}^d, \quad n = 1, 2, \dots \quad (6.6)$$

is dense in $(\mathbf{R}/\mathbf{Z})^d$.

This is exact again, for if the polynomials (6.5) are not $\mathbf{Q}$ -independent, i.e., some linear combination of them with rational coefficients has rational coefficients, then for

$$G_i(x_{i,1}, \dots, x_{i,\nu}) = w_{i,1}x_{i,1} + \dots + w_{i,\nu}x_{i,\nu}, \quad i = 1, \dots, d,$$

the sequence (6.6) is not dense in $(\mathbf{R}/\mathbf{Z})^d$ by Weyl's theorem.

Proof. If we set

$$F_i(x_{i,1}, \dots, x_{i,\nu}) = G_i(x_{i,1}, \dots, x_{i,\nu}) - (w_{i,1}x_{i,1} + \dots + w_{i,\nu}x_{i,\nu}),$$

then the claim follows from Theorem 3.7 exactly as we deduced Proposition 6.1 from Proposition 3.4. $\square$

6.2. Existence of distribution

We have seen that when considering the Weyl theory with periodic perturbations, we do not always get uniform distribution. This raises the question if in those cases the distribution of the sequences in question exist at all. We shall see that the answer depends on if we work on the torus or on $[0, 1)$ with fractional parts.

Let $X_1, X_2, \dots$ be a sequence on the torus $(\mathbf{R}/\mathbf{Z})^d$. We say that this sequence has distribution σ if for every continuous function $f : (\mathbf{R}/\mathbf{Z})^d \rightarrow \mathbf{R}$ we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(X_k) = \int f d\sigma. \quad (6.7)$$

An equivalent formulation is that for a dense set⁵ of $0 < a_i < b_i < 1$, $1 \leq i \leq d$, we have

$$\lim_{n \rightarrow \infty} \# \left\{ 1 \leq k \leq n \mid X_k \in \prod_{i=1}^d [a_i, b_i] \right\} = \sigma \left(\prod_{i=1}^d [a_i, b_i] \right). \quad (6.8)$$

⁵In fact, for all a_i, b_i for which the boundary of the box $\prod_{i=1}^d [a_i, b_i]$ has zero σ -measure.

Uniform distribution is when σ is the Haar measure on $(\mathbf{R}/\mathbf{Z})^d$ (which is the d -fold product of the normalized arc measure on the unit circle). In proving the existence of such a σ it is sufficient to show that the limit

$$L_f := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(X_k)$$

exists for all such f , for then L_f is clearly a positive linear functional and the Riesz representation theorem (cf. [4, Theorem 2.14]) gives (6.7) with some measure σ .

Now on the torus the existence of the distribution for polynomial sequences holds in all situations.

THEOREM 6.3. *Let $G_1, \dots, G_d$ be continuous mappings of $(\mathbf{R}/\mathbf{Z})^\nu$ into $\mathbf{R}/\mathbf{Z}$, and let $P_{i,j}$, $1 \leq i \leq d$, $1 \leq j \leq \nu$, be arbitrary polynomials. Then the sequence*

$$\left(G_1(P_{1,1}(n), \dots, P_{1,\nu}(n)), \dots, G_d(P_{d,1}(n), \dots, P_{d,\nu}(n)) \right)_{n=1}^\infty$$

of vectors has a distribution.

Proof. By introducing $(d-1)\nu$ dummy variables in each G_i (and hence replacing ν by νd) and by listing $P_{i,j}$ into a single sequence P_j it is sufficient to consider the case when $P_{i,j}$ does not depend on i , i.e., it is sufficient to consider sequences of the form

$$\left(G_1(P_1(n), \dots, P_\nu(n)), \dots, G_d(P_1(n), \dots, P_\nu(n)) \right)_{n=1}^\infty.$$

As before, we may assume that each P_j is without a constant term. Choose a maximal $\mathbf{Q}$ -independent subset of $P_1, \dots, P_\nu$. We may assume that this is $P_1, \dots, P_\mu$ for some μ , and first consider the case when $\mu \geq 1$. Every P_j , $\mu < j \leq \nu$, can be expressed as

$$P_j = R_j + \sum_{s=1}^{\mu} c_{s,j} P_s,$$

where the $c_{s,j}$ and the coefficients of the polynomial R_j are rational numbers. Choose an integer $q > 0$ such that every $qc_{s,j}$ is an integer, and all coefficients of all qR_j are integers. It is sufficient to show that the distribution of every subsequence

$$\left(G_1(P_1(k), \dots, P_\nu(k)), \dots, G_d(P_1(k), \dots, P_\nu(k)) \right)_{k=1, k \in \mathcal{N}_r}^\infty$$

exists, where $\mathcal{N}_r = \{k \mid k = ql + r\}_{l=1}^\infty$ and $0 \leq r < q$ is a fixed integer.

If $k \in \mathcal{N}_r$, then

$$P_j(k) = R_j(r) + \sum_{s=1}^{\mu} c_{s,j} P_s(k) \pmod{1},$$

so if

$$G_i^*(x_1, \dots, x_{\mu}) = G_i\left(x_1, \dots, x_{\mu}, R_{\mu+1}(r) + \sum_{s=1}^{\mu} c_{s,\mu+1} x_s, \dots, R_{\nu}(r) + \sum_{s=1}^{\mu} c_{s,\nu} x_s\right),$$

then

$$G_i(P_1(k), \dots, P_{\nu}(k)) = G_i^*(P_1(k), \dots, P_{\mu}(k)) \quad \text{for all } i.$$

Now if f is any continuous function on $(\mathbf{R}/\mathbf{Z})^d$ and we set

$$f^*(x_1, \dots, x_{\mu}) = f(G_1^*(x_1, \dots, x_{\mu}), \dots, G_d^*(x_1, \dots, x_{\mu})),$$

then we have to show the existence of the limit

$$\lim_{n \rightarrow \infty} \frac{q}{n} \sum_{k=1, k \in \mathcal{N}_r}^n f^*(P_1(k), \dots, P_{\mu}(k)).$$

Since, by Weyl's theorem, $\{(P_1(k), \dots, P_{\mu}(k))\}_{k=1, k \in \mathcal{N}_r}^{\infty}$ is uniformly distributed in $(\mathbf{R}/\mathbf{Z})^{\mu}$ (a consequence of the $\mathbf{Q}$ -independence of the polynomials $P_1, \dots, P_{\mu}$ ⁶), the preceding limit exists (and is actually $\int f^* d\sigma$ where σ is the Haar measure on $(\mathbf{R}/\mathbf{Z})^{\mu}$).

If $\mu = 0$, i.e., when all P_i have rational coefficients, then for a q for which all qP_j have integer coefficients and for $k = ql + r$ we have $P_j(k) = P_j(r) \pmod{1}$, i.e., $(P_1(k), \dots, P_{\nu}(k))$ is a constant vector in $(\mathbf{R}/\mathbf{Z})^{\nu}$ for $k \in \mathcal{N}_r$, from which the claim in the theorem immediately follows. $\square$

Finally, we show that the situation is different if we consider distribution not on the torus $\mathbf{R}/\mathbf{Z}$, but the distribution of the fractional parts (that belong to $[0, 1)$). The following proposition shows that in this case the distribution may not exist in very simple situations.

PROPOSITION 6.4. *Let α be irrational. Then there is a $1/\alpha$ -periodic continuous function f such that the distribution of the fractional parts $\{n\alpha + f(n)\}$, $n = 1, 2, \dots$, does not exist.*

The non-existence of the distribution in this case will mean that for all $b \in [7/8, 15/16]$ the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \#\{1 \leq k \leq n \mid \{k\alpha + f(k)\} \in [0, b]\}$$

does not exist.

⁶Note that any $\mathbf{Q}$ -dependence of $P_j(qx + r)$, $1 \leq j \leq \mu$, would automatically extend to a $\mathbf{Q}$ -dependence of $P_1, \dots, P_{\mu}$

Recall, however, that if f is β -periodic for some irrational β and $1, \alpha$ and β are rationally independent, then this phenomenon cannot happen since then the fractional parts $\{n\alpha + f(n)\}$, $n = 1, 2, \dots$, are uniformly distributed (see Proposition 4.1).

Proof of Proposition 6.4. It will be sufficient to construct a 1-periodic function h for which the sequence $\{n\alpha + h(n\alpha)\}$, $n = 1, 2, \dots$, of fractional parts has no distribution on $[0, 1)$, for then $f(x) := h(\alpha x)$ is suitable in Proposition 6.4.

Let $x_m = \{m\alpha\}$, and $S = \{x_m\}_{m=1}^\infty$. Let h_0 be the 1-periodic continuous function for which

$$h_0(x) = \begin{cases} -x & x \in [0, 1/2], \\ -1/2 + 2(x - 1/2), & x \in (1/2, 3/4], \\ 0, & x \in (3/4, 1). \end{cases}$$

Then

$$x + h_0(x) = \begin{cases} 0, & x \in [0, 1/2], \\ 3(x - 1/2), & x \in (1/2, 3/4], \\ x, & x \in (3/4, 1). \end{cases}$$

For $k = 0, 1, \dots$ we shall define a 1-periodic continuous function h_k and numbers $M_{k-1} < N_k < M_k$, where $-1/16 < h_{k+1} \leq h_k$ will be true for all k , and h_{k+1} will be obtained from h_k by modifying h_k on certain finitely many disjoint open intervals $(a_{k,s}, b_{k,s}) \subset (0, 1/2)$, $s \in L_k$, over which $h_k(x) = -x$. The modification will be done with the "triangle" function

$$\chi_{k,s}(x) = \begin{cases} a_{k,s} - x, & x \in (a_{k,s}, c_{k,s}], \\ x - b_{k,s}, & x \in (c_{k,s}, b_{k,s}), \\ 0, & \text{elsewhere in } [0, 1), \end{cases}$$

where $c_{k,s}$ is the midpoint of $(a_{k,s}, b_{k,s})$, i.e., we set

$$h_{k+1} = h_k + \sum_{s \in L_k} \chi_{k,s}(x), \quad (6.9)$$

and of course continue this h_{k+1} periodically with period 1. We shall also require that the endpoints $a_{k,s}, b_{k,s}$ do not belong to the set S , and $h_{k+1}(x) = h_0(x)$ outside a set consisting of finitely many closed intervals $\subset (0, 1/2)$ of total length $< 1/8$. Note that with such a construction the functions $\{h_k\}_{k=1}^\infty$ uniformly converge monotone decreasingly to a continuous function h such that $x+h(x)=x$ on $[3/4, 1)$, and for $x \in [0, 1)$ the fractional part $\{x + h(x)\}$ belongs to the interval $(7/8, 1)$ precisely if $x \in (7/8, 1)$ or if $x + h(x) < 0$, and in the latter case the fractional part $\{x + h(x)\}$ actually belongs to $(15/16, 1)$.

h_0 has already been given, and set $N_0 = 1$, $M_0 = 2$. Suppose h_k, N_k, M_k have already been defined with the property that $h_k(x) = h_0(x)$ outside a set of closed intervals $\subset (0, 1/2)$ of total length $< 1/8$, say of total length $1/8 - \delta_k$. The measure of the set

$$\{x \in [0, 1) \mid \{x + h_k(x)\} \in (7/8, 1)\} \quad (6.10)$$

is at most the measure of $(7/8, 1)$ plus the measure of those $x \in [0, 1/2]$, where $h_k(x) < h_0(x)$, hence (6.10) has measure $< 1/4$. Since h_k is 1-periodic, we have

$$\{m\alpha + h_k(m\alpha)\} = \{x_m + h_k(x_m)\},$$

and we get from the uniform distribution of the sequence $\{x_m\}_{m=1}^\infty$ in $[0, 1)$ that there is an $N_{k+1} > M_k$ such that

$$\#\{1 \leq m \leq N_{k+1} \mid x_m + h_k(x_m) \in (7/8, 1) \pmod{1}\} \leq \frac{N_{k+1}}{4}. \quad (6.11)$$

For the same reason there is an $M_{k+1} > N_{k+1}$ such that the set

$$L_k = \{N_{k+1} < m \leq M_{k+1} \mid x_m \in (0, 1/2), x_m + h_k(x_m) = 0\}$$

has at least

$$\#L_k \geq \left(\frac{1}{2} - \frac{1}{8}\right) M_{k+1} \quad (6.12)$$

elements. For each such x_m we have $h_k(x) = -x$ in a neighborhood of x_m (this follows from the fact that $a_{l,s}, b_{l,s} \notin S$ for $l < k$ by the induction hypothesis and by the fact that $h_k(x) < -x$ only on the unions of the intervals $(a_{l,s}, b_{l,s})$ with $l < k$), so we can choose for each $s \in L_k$ an interval $(a_{k,s}, b_{k,s}) \subset (0, 1/2)$ with center at x_s such that

- $a_{k,s}, b_{k,s} \notin S$,
- $(a_{k,s}, b_{k,s})$ does not contain any of the points x_m , $1 \leq m \leq N_{k+1}$,
- the total length of these $(a_{k,s}, b_{k,s})$, $s \in L_k$ is smaller than $\delta_k/16(k+1)$,
- $h_k(x) = -x$ on $(a_{k,s}, b_{k,s})$ and
- these intervals are pairwise disjoint for different s .

Finally, set h_{k+1} as in (6.9), and this finishes the induction.

The sequence $\{h_k\}_{k=1}^\infty$ converges decreasingly and uniformly to a 1-periodic continuous function h , and by the construction we have $h(x_m) = h_k(x_m)$ for all $1 \leq m \leq N_{k+1}$. Hence, (6.11) yields

$$\liminf_{k \rightarrow \infty} \frac{1}{N_{k+1}} \#\{1 \leq m \leq N_{k+1} \mid x_m + h(x_m) \in (7/8, 1) \pmod{1}\} \leq \frac{1}{4}. \quad (6.13)$$

On the other hand, for all $m \in L_k$ we have

$$\frac{1}{16} \leq x_m + h(x_m) \leq x_m + h_{k+1}(x_m) < 0,$$

and hence for all such m the fractional part $\{x_m + h(x_m)\}$ belongs to $(15/16, 1)$. But then (6.12) implies

$$\limsup_{k \rightarrow \infty} \frac{1}{M_{k+1}} \# \left\{ 1 \leq m \leq M_{k+1} \mid x_m + h(x_m) \in (15/16, 1) \pmod{1} \right\} \geq \frac{1}{2} - \frac{1}{8}. \quad (6.14)$$

Since $(15/16, 1)$ is a proper subinterval of $(7/8, 1)$, the relations (6.13) and (6.14) show that the sequence

$$\{x_m + h(x_m)\}_{m=1}^{\infty},$$

and hence also the sequence

$$\{\{m\alpha + h(m\alpha)\}\}_{m=1}^{\infty} \pmod{1}$$

has no distribution. □

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