engineering journals
Redefining Knowledge

uniform distribution theory

DOI: 10.66833/UDT-2026-0004
Unif. Distrib. Theory 21 (2026), no.1, 1-14

Binary-ternary collisions and the last significant digit of $n!$ in base 12

Jean-Marc Deshouillers, Pascal Jelinek and Lukas Spiegelhofer

ABSTRACT. The third-named author recently proved [Israel J. of Math. 258 (2023), 475–502] that there are infinitely many collisions of the base-2 and base-3 sum-of-digits functions. In other words, the equation

$$s_2(n) = s_3(n)$$

admits infinitely many solutions in natural numbers. We refine this result and prove that every integer a in $\{1,2,\dots,11\}$ appears as the last nonzero digit of $n!$ in base 12 infinitely often.

Communicated by Yann Bugeaud

1 Introduction

The sequence A136698 of the On-Line Encyclopedia of Integer Sequences [8] provides the last nonzero digit of the number $n!$ in base 12.

Let us define this notion. Writing a positive integer n in base b (where $b \geq 2$ is an integer) as

$$n = \sum_{j\geq 0} a_j^{(b)}(n) b^j \quad \text{where} \quad a_j^{(b)}(n) \in \mathcal{D}_b = \{0,1,\dots,b-1\}, \tag{1.1}$$

its *last nonzero digit in base* b is defined by

$$\ell_b(n) = \min\{j : a_j^{(b)}(n) \neq 0\}. \tag{1.2}$$

A quick glance at the first elements of the above-mentioned sequence reveals that all the numbers in $\mathcal{D}_{12}^* = \{1,2,\dots,11\}$ occur in the sequence, the numbers 4 and 8 seeming to occur more frequently.

2020 Mathematics Subject Classification: 11A63, 11K31, 11B75.
Keywords: Sums of digits in bases 2 and 3, last significant digit of n!

Indeed, it was proved by I. Z. Ruzsa and the first-named author in 2011 [5] that the digits 4 and 8 each occur with asymptotic density 1/2. The next year, the first-named author showed in [3] that each of the digits 3, 6 and 9 occurs infinitely often. This latter paper concludes with two open questions. (1) Do all the numbers from $\{1,2,\dots,10,11\}$ occur infinitely often in the sequence $(\ell_{12}(n!))_n$? (2) How often do 3,6 and 9 occur in $(\ell_{12}(n!))_{n\le N}$ when N is large? The first aim of this paper is to give answers to these questions.

Theorem 1.1. *There exist real numbers δ,η, satisfying $0<\delta\le\eta<1$, such that for each integer a in $\{1,2,3,5,6,7,9,10,11\}$, we have*

$$N^{\delta} \ll \mathrm{card}\{n\le N:\ell_{12}(n!)=a\} \ll N^{\eta}. \tag{1.3}$$

As we shall explain in Section 3.1, our question is closely connected to the existence of integers n such that the sums of their digits in bases 2 and 3 are equal. Here the *sum of digits* of the integer n in base b is given by

$$s_b(n)=\sum_{j\ge 0} a_j^{(b)}(n), \tag{1.4}$$

where $a_j^{(b)}(n)$ is introduced in (1.1). The third-named author [10] has recently proved that there exist infinitely many integers n such that $s_2(n)=s_3(n)$. Our proof of Theorem 1.1 relies on an extension of this result.

Theorem 1.2. *Let k,ℓ,a be nonnegative integers. There exists a positive real number β such that*

$$\mathrm{card}\{n\le N:n\equiv a(\mathrm{mod}\,2^k3^{\ell}) \text{ and } s_2(n)=s_3(n)\}\gg N^{\beta}, \tag{1.5}$$

where the implied constant may depend on k and ℓ, while β is independent of k,ℓ,a.

We shall conclude the paper by some consideration on the question of characterizing integers b such that for any $a\in\{1,2,\dots,b-1\}$, we have $\ell_b(n!)=a$ infinitely often.

2 Acknowledgements

Jean-Marc Deshouillers acknowledges support by the FWF–ANR project ArithRand (grant numbers I4945 and ANR-20-CE91-0006); Pascal Jelinek was supported by the FWF project P36137 and the FWF–RSF joint project DIOARIANA (grant number I5554); Lukas Spiegelhofer acknowledges support by the FWF–ANR projects ArithRand (grant numbers I4945 and ANR-20-CE91-006) and SymDynAr (grant numbers I6750 and ANR-23-CE40-0024), and by the FWF project P36137.

3 Proof of Theorem 1.1 assuming Theorem 1.2

3.1 Valuation and last nonzero digit

Let b be an integer larger than 1 and n a positive integer. We let $v_b(n)$ denote the largest nonnegative integer such that $b^{v_b(n)}$ divides n, and we write

$$n = b^{v_b(n)} n_b, \quad \text{where } b \text{ does not divide } n_b. \tag{3.1}$$

In the case when b is the product of two coprime integers both larger than 1, say $b = b_1 b_2$, the following lemma gives us the connection between the divisibility of $\ell_b(n)$ by b_1 and the comparison of $v_{b_1}(n)$ and $v_{b_2}(n)$.

Lemma 3.1. *Let $b = b_1 b_2$, where b_1 and b_2 are coprime and larger than 1. Then*

$$v_{b_1}(n) > v_{b_2}(n) \Leftrightarrow b_1 \mid \ell_b(n).$$

Proof. The coprimality of b_1 and b_2 implies

$$v_{b_1 b_2}(n) = \min\big(v_{b_1}(n), v_{b_2}(n)\big). \tag{3.2}$$

For any base c, by the definition of $\ell_c(n)$, we have

$$n \equiv c^{v_c(n)} \ell_c(n) (\mathrm{mod} c^{v_c(n)+1}). \tag{3.3}$$

Assume that we have $v_{b_1}(n) > v_{b_2}(n)$. By (3.2), we have $v_b(n) = v_{b_2}(n)$ and by (3.3) we thus have

$$n \equiv b^{v_{b_2}(n)} \ell_b(n) (\mathrm{mod} b^{v_{b_2}(n)+1})$$

which implies

$$n \equiv b_1^{v_{b_2}(n)} \ell_b(n) \left(\mathrm{mod} b_1^{v_{b_2}(n)+1}\right).$$

But $b_1^{v_{b_1}(n)}$ divides n and so $b_1^{v_{b_2}(n)+1}$ also divides n, so that we have

$$b_1^{v_{b_2}(n)} \ell_b(n) \equiv 0 \left(\mathrm{mod} b_1^{v_{b_2}(n)+1}\right)$$

and thus $\ell_b(n) \equiv 0 (\mathrm{mod} b_1)$.

Concerning the opposite direction, we assume that b_1 divides $\ell_b(n)$. From (3.3) applied with $c = b$ we deduce that n is divisible by $b_1^{v_b(n)+1}$, therefore

$$v_{b_1}(n) \geq v_b(n) + 1 = \min(v_{b_1}(n), v_{b_2}(n)) + 1,$$

which finally implies $v_{b_1}(n) > v_{b_2}(n)$.

In the case when $b = 12 = 4 \cdot 3 = 3 \cdot 4$, this leads to the following corollary, where (3.4) and (3.5) follow directly from Lemma 3.1, and (3.6) is an easy consequence of the other two.

Corollary 3.2. *We have*

$$v_4(n) > v_3(n) \Leftrightarrow \ell_{12}(n) \in \{4,8\}, \tag{3.4}$$

$$v_4(n) < v_3(n) \Leftrightarrow \ell_{12}(n) \in \{3,6,9\}, \tag{3.5}$$

$$v_4(n) = v_3(n) \Leftrightarrow \ell_{12}(n) \in \{1,2,5,7,10,11\}. \tag{3.6}$$

We owe to Legendre a compact formula for the valuation of $n!$ in a prime base p, namely

$$v_p(n!) = \frac{n - s_p(n)}{p-1} \tag{3.7}$$

where the sum of digits function s_p is defined in (1.4).

This implies that for a prime power base p^a we have

$$v_{p^a}(n!) = \left\lfloor \frac{n - s_p(n)}{a(p-1)} \right\rfloor. \tag{3.8}$$

We shall refer to (3.8) as “Legendre’s formula".

In the special case of the base 12, we have

$$v_3(n!) = \frac{n - s_3(n)}{2} \text{ and } v_4(n!) = \left\lfloor \frac{n - s_2(n)}{2} \right\rfloor.$$

Noticing that $v_3(n!)$ is an integer — which is equivalent to $s_3(n) \equiv n \pmod 2$ — we have $v_3(n!) = v_4(n!)$ if and only if $s_3(n)$ is in $\{s_2(n), s_2(n)+1\}$. Thus, by (3.6), we have

$$\ell_{12}(n!) \in \{1,2,5,7,10,11\} \Leftrightarrow s_3(n) \in \{s_2(n), s_2(n)+1\}. \tag{3.9}$$

We can go one step further and show that

$$\ell_{12}(n!) \in \{1,5,7,11\} \Leftrightarrow s_3(n) = s_2(n) \tag{3.10}$$

and

$$\ell_{12}(n!) \in \{2,10\} \Leftrightarrow s_3(n) = s_2(n) + 1. \tag{3.11}$$

Let us prove it. We first assume that $s_2(n) = s_3(n)$. By Legendre’s formula, $n!$ can be written as $3^u R$ and $4^u S$, where R is not divisible by 3 and S is not divisible by 2. This implies $n! = 12^u T$, where T is not divisible by 2 or 3, therefore $\ell_{12}(n!)$ belongs to $\{1,5,7,11\}$.

From this the implication "⇐" in (3.10) follows. In a similar manner, when $s_2(n) = s_3(n) - 1$, we can write $n! = 12^u T$ for some T which is divisble by 2, but neither divisible by 4 nor by 3. Thus $\ell_{12}(n!)$ belongs to $\{2,10\}$, which implies the implication "⇐" in (3.11). These two relations, combined with the equivalence (3.9), imply the equivalences in (3.10) and (3.11).

3.2 Reduction to consecutive blocks

From (3.9), the main result of [10] implies that at least one of the digits 1,2,5,7,10,11 occurs quite often, but it may be difficult to determine the digits occurring sufficiently often. To show that indeed all of them occur sufficiently often, we shall use a trick introduced in [3], considering blocks of consecutive integers in which all those values occur.

Proposition 3.3. *If n is divisible by 12^3 and $s_2(n) = s_3(n)$, then*

$$\{1,2,5,7,10,11\} \subset \{\ell_{12}((n+1)!), \ell_{12}((n+2)!), \dots, \ell_{12}((n+10)!))\}.$$

Proof. The statement is a simple consequence of the following claims.

Claim 3.4. *We have*

$$(\ell_{12}(1!), \ell_{12}(2!), \ell_{12}(5!), \ell_{12}(6!), \ell_{12}(7!), \ell_{12}(10!)) = (1,2,10,5,11,7).$$

Proof. This is a simple computation; one can also look at the sequence A136698 in [8].

Claim 3.5. *If n is divisible by 12^3 and k is an integer in $[1,10]$, we have*

$$\ell_{12}((n+1)(n+2)\cdots(n+k)) = \ell_{12}(k!). \tag{3.12}$$

Proof. Since n is divisible by 12^3, we have

$$(n+1)(n+2)\cdots(n+k) \equiv k! \,(\mathrm{mod} 12^3).$$

For $k \leq 8$, the claim follows from the fact that $8!$ is not divisible by 12^3.

For k in $\{9,10\}$, we go one step further in the expansion of the product as a polynomial in n, keeping the first-degree terms in n, i.e.

$$(n+1)(n+2)\cdots(n+k) \equiv k! + n\left(\frac{k!}{1} + \cdots + \frac{k!}{k}\right) (\mathrm{mod} 12^6).$$

We note that $k!$ is not divisible by 12^5, but all the terms inside the parentheses in the RHS are divisible by 12^2.

Claim 3.6. *Let x and y be two integers such that $\ell_{12}(x)$ is in $\{1,5,7,11\}$. We have*

$$\ell_{12}(xy) \equiv \ell_{12}(x)\ell_{12}(y)(\mathrm{mod}12). \tag{3.13}$$

Proof. We can write $x = 12^{\alpha}(\ell_{12}(x) + 12X)$ and $y = 12^{\beta}(\ell_{12}(y) + 12Y)$. Thus, we have $xy = 12^{\alpha+\beta}(\ell_{12}(x)\ell_{12}(y) + 12Z)$. Since $\ell_{12}(y)$ is not zero and $\ell_{12}(x)$ is invertible modulo 12, the product $\ell_{12}(x)\ell_{12}(y)$ is not congruent to 0 modulo 12, and thus congruent to $\ell_{12}(xy)$ modulo 12.

Claim 3.7. *The set* $\{1,2,10,5,7,11\}$ *of residues modulo* 12 *is invariant under multiplication by* 1, 5, 7 *or* 11.

Proof. A simple computation shows that we respectively get the sets $\{1,2,10,5,7,11\}$, $\{5,10,2,1,11,7\}$, $\{7,2,10,11,1,5\}$ and $\{11,10,2,7,5,1\}$.

Let n satisfy the conditions of Proposition 3.3, and suppose that $k \in \{1,2,5,6,7,10\}$. By (3.10), $\ell_{12}(n!)$ is in $\{1,5,7,11\}$. By Claim 3.6, we have $\ell_{12}((n+k)!) \equiv \ell_{12}(n!)\ell_{12}((n+1)(n+2)\cdots(n+k))(\mathrm{mod}12)$. Claim 3.5 implies $\ell_{12}((n+k)!) \equiv \ell_{12}(n!)\ell_{12}(k!)(\mathrm{mod}12)$. Moreover, by (3.10) and Claim 3.4, the set $\{\ell_{12}((n+1)!), \ell_{12}((n+2)!), \ell_{12}((n+5)!), \ell_{12}((n+6)!), \ell_{12}((n+7)!), \ell_{12}((n+10)!)\}$ is — modulo 12 — the product by 1, 5, 7 or 11 of the set $\{1,2,10,5,7,11\}$. Claim 3.7 implies that it is equal to the set $\{1,2,5,7,10,11\}$. This ends the proof of Proposition 3.3.

3.3 Proof of Theorem 1.1

We start with the lower bound. Since we are concerned with finitely many values of a, it is enough to prove that for each value of a there exists δ such that the lower bound of (1.3) holds.

For a in $\{1,2,5,7,10,11\}$, this is a simple consequence of Proposition 3.3 and Theorem 1.2, applied with $a = 0, k = 6$, and $\ell = 3$.

Let us now assume that $a \in \{3,6,9\}$. The following result is Proposition 3 in the article [3].

Proposition 3.8. *Let* n *be divisible by* 144, *and such that* $v_3(n!) \geq v_4(n!) + 2$. *Then for any* $a \in \{3,6,9\}$ *there exists* $k \in \{0,2,3,7\}$ *such that* $\ell_{12}((n+k)!) = a$.

The number $R = 3^9 2^4$ was introduced in [3]. It satisfies the properties $144 \parallel R, s_2(R) = 8, s_3(R) = 4, R < 2^{19}$ and $R < 3^{12}$.

By Theorem 1.2, there are — up to constant — more than N^{β} integers $n \leq N$ divisible by $2^{19}3^{12}$, and such that $s_2(n) = s_3(n)$.

For such n, one has $s_2(n+R) = s_2(n) + 8$ and $s_3(n+R) = s_3(n) + 4 = s_2(n) + 4 = s_2(n+R) - 4$. By Legendre's formula, we thus have $v_3((n+R)!) \geq v_4((n+R)!) + 2$. By construction, $n+R$ is divisible by 144, and by Proposition 3.8, each of the digits $3,6$ and 9 occur in the sequence $\{\ell_{12}((n+R)!), \ell_{12}((n+R+2)!), \ell_{12}((n+R+3)!), \ell_{12}((n+R+7)!)\}$.

As regards the upper bound, the following result was stated in [5], but not proved. The authors simply used a result of [1], later refined in [2], leading to the upper bound $o(x)$.

Claim 3.9. *We can show that* $card\{n \leq x : \ell_{12}(n!) = a\} = O(x^c)$ *with some* $c < 1$ *whenever* $a \neq 4,8$.

We give here a proof of this claim, based on the following result, for which we give a short proof relying on Hoeffding's inequality. A similar result was established in [4], using multivariate distribution and Stirling's formula.

Lemma 3.10. *Let* $b \geq 2$ *be an integer and* δ *a positive real number. There exists a real number* $c < 1$ *such that for all positive real* x *one has*

$$\operatorname{card}\left\{0 < n < x : \left|s_b(n) - \frac{b-1}{2\log b}\log n\right| \geq \delta \log n\right\} \ll x^c. \tag{3.14}$$

Proof. It is enough to prove the lemma when x is a sufficiently large power of b, say $x = b^K$. A little computation shows that we can find λ and μ with $0 < \lambda < \mu < 1$ such that for any k in $[\lambda K, \mu K]$, one has

$$(K-k)(\delta \log b + (b-1)/2) \leq (\delta/2)\log b^K.$$

Then, for n in $[b^k, b^K)$, one has

$$\left|s_b(n) - \frac{b-1}{2\log b}\log n\right| \geq \delta \log n \;\Rightarrow\; \left|s_b(n) - \frac{b-1}{2}K\right| \geq (\delta/2)\log b \times K. \tag{3.15}$$

There are $(b^K)^{k/K}$ integers less than b^k and $k/K \leq \mu < 1$. Using Equation (3.15), we see that in order to prove Lemma 3.10 (3.14), it is enough to prove the following statement.

For any positive η, there exists a real number $c < 1$ such that

$$\operatorname{card}\left\{0 \leq n < b^K : \left|s_b(n) - \frac{b-1}{2}K\right| \geq \eta K\right\} \ll b^{cK}, \quad \text{as } K \to \infty. \tag{3.16}$$

We can rephrase this question in a probabilistic setting. Let $\xi_1, \cdots, \xi_K$ be independent random variables uniformly distributed on $\{0,1,\cdots, b-2, b-1\}$, and let $S_K = \xi_1 + \cdots + \xi_K$. We readily see that S_K has the following two properties.

For any integer m:

$$\mathbb{P}\{S_K = m\} = b^{-K}\operatorname{card}\{n < b^K : s_b(n) = m\}, \tag{3.17}$$

$$\mathbb{E}(S_K) = K\frac{b-1}{2}. \tag{3.18}$$

Hoeffding's inequality [7] gives the upper bound

$$\mathbb{P}\{|S_K - \mathbb{E}(S_K)| \geq t\} \leq 2\exp\left(-\frac{2t^2}{K(b-1)^2}\right) \tag{3.19}$$

for the tail of the distribution of S_K. Relations (3.17), (3.18), and (3.19) imply

$$\begin{aligned} \operatorname{card}\left\{0 \leq n < b^K : \left|s_b(n) - \frac{b-1}{2}K\right| \geq \eta K\right\} &= b^K \mathbb{P}\{|S_K - \mathbb{E}(S_K)| \geq \eta K\} \\ &\leq 2b^K \exp\left(-\frac{2(\eta K)^2}{K(b-1)^2}\right) \\ &\leq 2b^{cK}, \end{aligned}$$

where

$$c = 1 - \left(\frac{2\eta^2}{(b-1)^2 \log b}\right).$$

This proves the validity of (3.16) and thus completes the proof of Lemma 3.10.

By Corollary 3.2, $\ell_{12}(n!) \notin \{4,8\}$ is equivalent to $v_4(n!) \leq v_3(n!)$, and thus (3.7) implies that $\ell_{12}(n!) \notin \{4,8\}$ is equivalent to $s_3(n) \leq s_2(n) + 1$. This relation occurs only rarely:

- either $s_3(n) \leq 0.82 \log n$. Since $(3-1)/(2\log 3) = 0.91\ldots$, the set of such integers n has an exponential density less than 1 by Lemma 3.10, applied with $b = 3$,

- or $s_3(n) > 0.82 \log n$. We have then $s_2(n) \geq s_3(n) - 1 > 0.82 \log n - 1$. Since $(2-1)/(2\log 2) = 0.72\ldots$, the set of such integers n has an exponential density less than 1 by Lemma 3.10, applied with $b = 2$.

This completes the proof of Claim 3.9, as well as that of Theorem 1.1.

4 Proof of Theorem 1.2

4.1 Revisiting the collisions-paper

In this section, we indicate how to obtain a complete proof of Theorem 1.2. This is accomplished by minimalistic changes of the paper [10], which we indicate in the sequel.

First, we want to find *almost-collisions* in a given residue class $a + 2^k 3^\ell \mathbb{N}$, replacing the statement "$s_2(n) - s_3(n) \in \{0,1\}$ for infinitely many $n \in 9 + 12\mathbb{N}$". For this, we modify the definition of η ([10], Equation (12)) by choosing $m = \max\{2, k\}$, and $\eta := 2^m \lfloor \eta_0 2^{-m} \rfloor$.

In ([10], Proposition 2.2), the definition of the shifts d_j has to be changed in order to ensure that we do not leave our prescribed residue class modulo $2^k 3^\ell$. For this, let

$$d_j := (1^{(j+1+J)\eta} 0^\ell)_3,$$

which is obviously divisible by 3^ℓ. Moreover, the fact

$$2^m \mid \sum_{0 \le j < R \cdot 2^m} 3^j$$

(use a geometric series and Euler–Fermat) implies $2^k \mid d_j$. Moreover, we ask for the existence of $L \in \{0, \dots, 2^\nu 3^\beta - 1\}$ such that $L \equiv a \bmod 2^k 3^\ell \mathbb{N}$, instead of $L \equiv 9 \bmod 12$. The rest of the statement is unchanged. In the proof, "blocks of 1s of length η" appear. Their length is a multiple of four also in the present paper, and the three addition patterns depicted below ([10], Equation (42)) can be reused in order to obtain an "almost-collision" in the sense $s_2(n) - s_3(n) \in \{0,1\}$.

In ([10], Equation (35)), we replace the first line by a congruence $a \equiv r \bmod 2^m$, where r is arbitrary. Also, the construction of the integers k_j, beginning with ([10], Equation (43)), has to account for an arbitrary residue class modulo 3^ℓ. This flexibility is needed for obtaining collisions in *any* residue class modulo $2^k 3^\ell$: by intersection of residue classes such as in ([10], Equation (41)), and the Chinese remainder theorem, we can obtain any prescribed L.

Reusing ([10], Propositions 2.3 and 2.4), we obtain the following statement.

Proposition 4.1. *Let $k, \ell \ge 0$ be integers, and $0 \le L < 2^k 3^\ell$. There exist infinitely many positive integers $n \in L + 2^k 3^\ell \mathbb{N}$ such that*

$$s_2(n) - s_3(n) \in \{0,1\}.$$

4.2 Removal of almost collisions

In the previous subsection we have shown that we can achieve almost-collisions in any residue class $L \bmod 2^k 3^\ell$. We now want to remove the inelegant deviation by 1 from a true collision, while remaining in the residue class. The following proposition states that we can move to a sparser arithmetic progression $L' \bmod 2^{k'} 3^{\ell'}$ in order to change an almost collision into a proper collision, hence proving Theorem 1.2.

Proposition 4.2. *Let $2 \le p < q$ be coprime integers. Let a_0 be a non-negative integer, say it is smaller than $p^{y_{p_0}} q^{y_{q_0}}$, for some integers p_0 and q_0. Let d_i, m_i, M_i be integers for $1 \le i \le \ell$. Then the following statements are equivalent.*

(i) Each d_i is divisible by $gcd(p-1, q-1)$.
(ii) There exist integers $t_1, \dots, t_\ell, y_p > y_{p_0}, y_q > y_{q_0}$, and a, satisfying

- *$t_i \equiv m_i \bmod M_i$ for all $1 \le i \le \ell$, and*
- *$a \equiv a_0 \bmod p^{y_{p_0}} q^{y_{q_0}}$*
- *for all $n \equiv a \bmod p^{y_p} q^{y_q}$ and for all $1 \le i \le \ell$,*

$$s_p(n + t_i) - s_q(n + t_i) = s_p(n) - s_q(n) + d_i\,. \tag{4.1}$$

Before we can prove this proposition, we need the following lemma.

Lemma 4.3. *Let y and z be positive integers, $m', r \in [0, p^y)$ integers, and $c \in \{1,2,\dots,p-1\}$. Let $m = cp^y + m'$ and n be a positive integer congruent to $cp^{y+z} - cp^y + r$ modulo p^{y+z+1}. We have*

$$s_p(m+n) = \begin{cases} s_p(m+r) + s_p(n-r) - z(p-1), & \text{if } m + r < p^{y+1}; \\ s_p(m+r) + s_p(n-r) - (z-1)(p-1), & \text{if } m + r \ge p^{y+1}. \end{cases} \tag{4.2}$$

Proof. Let us write

$$\begin{array}{rcccccccccccccc} m &=& 0 & p^{y+z+1} &+& 0 & p^{y+z} &+& 0 & p^{y+z-1} &+ \cdots +& c & p^y &+& m' \\ n &=& n' & p^{y+z+1} &+& (c-1) & p^{y+z} &+& (p-1) & p^{y+z-1} &+ \cdots +& (p-c) & p^y &+& r. \end{array}$$

We have

$$s_p(n-r) = s_p(n') + (c-1) + (z-1)(p-1) + (p-c) = s_p(n') + z(p-1). \tag{4.3}$$

When m and n are added, the "p^y-terms" always produce a carry, which propagates up to the "p^{y+z}-terms", but no further, since $c - 1 \le p - 2$.

Next, we need to reformulate the conditions $m + r < p^{y+1}$ and $m + r \ge p^{y+1}$ in (4.2) into statements involving only m', r, c. We observe that

$$m + r < p^{y+1} \Leftrightarrow m' + r < p^y \text{ or } c < p - 1; \tag{4.4}$$

$$m + r \ge p^{y+1} \Leftrightarrow m' + r \ge p^y \text{ and } c = p - 1. \tag{4.5}$$

We now make the case distinction depending on the values of m', r, c.

If $m' + r < p^y$, then

$$s_p(m+n) = s_p(n') + c + z \times 0 + s_p(m'+r) \text{ and } s_p(m+r) = c + s_p(m'+r). \tag{4.6}$$

Relations (4.3) and (4.6) lead to the first part of the first case of (4.2).

If $m' + r \geq p^y$, then we can write $m' + r = p^y + (m' + r - p^y)$; since both m' and r are less than p^y, we have $m' + r - p^y < p^y$ and thus

$$s_p(m+n) = s_p(n') + c + (z-1) \times 0 + 1 + s_p(m'+r-p^y) = s_p(n') + c + s_p(m'+r). \tag{4.7}$$

But, when adding m and r, we get $m + r = cp^y + m' + r = (c+1)p^y + (m' + r - p^y)$, which leads to

$$s_p(m+r) = (c+1) + \big(s_p(m'+r) - 1\big) = c + s_p(m'+r) \text{ if } c < p-1 \tag{4.8}$$

$$s_p(m+r) = 1 + \big(s_p(m'+r) - 1\big) = s_p(m'+r) \text{ if } c = p-1. \tag{4.9}$$

Equations (4.3), (4.4), (4.7) and (4.8) lead to the second part of the first case of (4.2), whereas Equations (4.3), (4.5), (4.7) and (4.9) lead to the second case of (4.2).

Now we are able to prove Proposition 4.2.

Proof. First, we will show that (ii) implies (i). It is known that $s_p(n) \equiv n \bmod p-1$, and $s_q(n) \equiv n \bmod q-1$. Hence evaluating equation (4.1) modulo $\gcd(p-1, q-1)$ we get

$$n + t_i - (n + t_i) \equiv n - n + d_i \ .$$

Hence $d_i \equiv 0 \bmod \gcd(p-1, q-1)$.

Now we will prove that (i) implies (ii). We will give a recursive algorithm that gives us t_1 up to t_ℓ, and the corresponding residue class a given any a_0. Without loss of generality, we can assume that $a_0 < p^{y_{p_0}} q^{y_{q_0}}$. Pick $t_1 > p^{y_{p_0}+1} q^{y_{q_0}+1}$ such that $t_1 \equiv m_1 \bmod M_1$. By Bezout's lemma (also contained in Euclid's work, see [6]) we can find integers z'_{p_1} and z'_{q_1} such that

$$-d_1 + s_p(t_1 + a_0) - s_q(t_1 + a_0) - s_p(a_0) + s_q(a_0) = z'_{p_1}(p-1) - z'_{q_1}(q-1)\,.$$

Let c_{p_1} be the leading digit of t_1 in base p, say it is the coefficient of $p^{\tilde{y}_{p_1}}$, and let c_{q_1} be the leading digit of t_1 in base q, say it is the coefficient of $q^{\tilde{y}_{q_1}}$. Additionally, let $z_{p_1} = z'_{p_1}$, if $t_1 + a_0 < p^{\tilde{y}_{p_1}+1}$, and $z_{p_i} = z'_{p_1} + 1$, if $t_1 + a_0 \geq p^{\tilde{y}_{p_1}+1}$. Define z_{q_1} analogously.

Now pick n such that it satisfies the congruences

$$n \equiv c_{p_1} p^{\tilde{y}_{p_1}+z_{p_1}} - c_{p_1} p^{\tilde{y}_{p_1}} + a_0 mod p^{\tilde{y}_{p_1}+z_{p_1}+1}, \tag{4.10}$$

$$n \equiv c_{q_1} q^{\tilde{y}_{q_1}+z_{q_1}} - c_{q_1} q^{\tilde{y}_{q_1}} + a_0 mod q^{\tilde{y}_{q_1}+z_{q_1}+1} \tag{4.11}$$

Let the residue class of $n \bmod p^{\tilde{y}_{p_1}+z_{p_1}+1}q^{\tilde{y}_{q_1}+z_{q_1}+1}$ be called a_1. Also, we write $y_{p_1} = \tilde{y}_{p_1} + z_{p_1} + 1$ and $y_{q_1} = \tilde{y}_{q_1} + z_{q_1} + 1$. Using Lemma 4.3 twice, once with $c = c_{p_1}, y = \tilde{y}_{p_1}, r = a_0, z = z_{p_1}$, and once with $c = c_{q_1}, y = \tilde{y}_{q_1}, r = a_0, z = z_{q_1}$, we obtain

$$\begin{aligned} s_p(n + t_1) &= s_p(t_1 + a_0) + s_p(n - a_0) - z'_{p_1}(p - 1), \\ s_q(n + t_1) &= s_q(t_1 + a_0) + s_q(n - a_0) - z'_{q_1}(q - 1). \end{aligned} \tag{4.12}$$

Subtracting the second line from the first, we obtain

$$s_p(n + t_1) - s_q(n + t_1) = s_p(n - a_0) + s_p(a_0) - s_q(n - a_0) - s_q(a_0) + d_1 \tag{4.13}$$

$$s_p(n + t_1) - s_q(n + t_1) = s_p(n) - s_q(n) + d_1 \tag{4.14}$$

by our choices of z'_{p_1} and z'_{q_1}, since a_0 are the last y_p or y_q digits of n, respectively. Now we will repeat this process another $\ell - 1$ times, and during each iteration, we increase the indices of each variable by 1 compared to the previous iteration. Since in the k-th iteration, we have not changed the last $y_{p_{k-1}}$ or $y_{q_{k-1}}$ digits of n, respectively, n is still in the same residue class $\bmod p^{y_{p_{k-1}}}q^{y_{q_{k-1}}}$. Hence the integers $t_1, \dots, t_{k-1}$ still satisfy their respective equation, for n given in the new residue class. Taking a to be a_ℓ completes the proof.

5 A general question

By the main result of [5] and Theorem 1.1, the base $b = 12$ has the following property

Property 5.1. *(Full-range condition).* $\forall a \in \{1,2, \dots, b-1\}, \exists_\infty n: \ell_b(n!) = a$.

Thanks to B. Sobolewski [9], we know that all bases that are prime powers also satisfy Property 5.1. It would be interesting to characterize all integers b satisfying the *full-range condition.*

The following straightforward claim leads to a necessary condition for a base b to satisfy Property 5.1.

Claim 5.2. *Let b be a natural integer larger than 2 and $b = p_1^{a_1} \cdots p_s^{a_s}$ a decomposition of b as a product of pairwise coprime prime powers. A necessary condition that at least one integer a coprime with b occurs infinitely many times in the sequence $(\ell_b(n!))_n$ is*

$$a_1(p_1 - 1) = \cdots = a_s(p_s - 1). \tag{5.1}$$

Proof. If the condition (5.1) is not satisfied, then $s \geq 2$ and there exist two indices i and j such that $a_i(p_i - 1) < a_j(p_j - 1)$. By (3.8) and the fact that $s_p(n) = O(\log n)$, we have, for sufficiently large n : $v_{p_j}(n!) < v_{p_i}(n!)$ and so p_i divides $\ell_b(n!)$.

By the same reasoning, we easily prove the following claim.

Claim 5.3. *Let b be a natural integer larger than 2 and $b = p_1^{a_1} \cdots p_s^{a_s}$ a decomposition of b as a product of pairwise coprime prime powers. A necessary and sufficient condition that at least one integer a coprime with b occurs infinitely many times in the sequence $(\ell_b(n!))_n$ is*

$$a_1(p_1 - 1) = \cdots = a_s(p_s - 1) \text{ and } \exists_\infty n: s_{p_1}(n) = \cdots = s_{p_s}(n). \qquad (5.2)$$

We believe that Claim 5.3 is also necessary and sufficient for b to satisfy the *full-range condition* (Property 5.1).

In a letter to the Number Theory List [11], Zhi-Wei Sun mentions the following

Conjecture 5.4. *(Shawkwei Moh, 1990). For any k distinct primes $p_1, \ldots, p_k$, there are infinitely many positive integers n such that*

$$v_{p_1}(n!): v_{p_2}(n!): \cdots : v_{p_k}(n!) = \frac{1}{p_1-1} : \frac{1}{p_2-1} : \cdots : \frac{1}{p_k-1}. \qquad (5.3)$$

This conjecture would imply that Conditions (5.1) and (5.2) are equivalent. We have however some heuristic reasons, which we shall present elsewhere, to doubt that this is the case.

REFERENCES

[1] Bassily, Nader L. -*Distribution of q-ary digits in some sequences of integers*, Ann. Univ. Sci. Budapest, Sect. Comput. 14 (1994), 13-22.

[2] Bassily, Nader L. and Kátai, Imre. -*Distribution of the values of q-additive functions on polynomial sequences*, Acta Math. Hung. 68 (1995), 353-361.

[3] Deshouillers, Jean-Marc. - *A footnote to "The least non zero digit of $n!$ in base 12"*, Unif. Distrib. Theory, 7 (2012), 71-73.

[4] Deshouillers, Jean-Marc, Habsieger, Laurent, Laishram, Shanta and Landreau, Bernard. -*Sums of the digits in Bases 2 and 3*. Festschrift in Honour of Robert Tichy's 60th Birthday, Springer (2017), 211-217.

[5] Deshouillers, Jean-Marc and Ruzsa, Imre. - *The least nonzero digit of $n!$ in base 12*, Publ. Math. Debrecen, 79 (2011), 163-167.

[6] Granville, Andrew - *It is not "Bézout's identity"*, Preprint, 2024 (arXiv:2406.15642).

[7] Hoeffding, Wassily. - *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. 58 (1963), 13-30.

[8] Sloane, Neil J. A. - *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org.

[9] Sobolewski, Bartosz .- *On the last nonzero digits of* $n!$ *in a given base*, Acta Arith. 191 (2019), 173-189.

[10] Spiegelhofer, Lukas. - *Collisions of digit sums in bases* 2 *and* 3, Israel J. of Math. 258 (2023), 475-502.

[11] Sun, Zhi-Wei. - *A conjecture of Shawkwein Moh*, a letter sent to nmbrthry@listserv.nodak.edu on March 11th, 2012.

Received December 11, 2024
Accepted May 19, 2026
Published June 1, 2026

Jean-Marc Deshouillers

Institut de Mathématiques de Bordeaux, Universit´e de Bordeaux, Bordeaux INP and CNRS, 33405 Talence, France.

jean-marc.deshouillers@math.u-bordeaux.fr

ORCID iD: 0000-0002-1826-689X

Pascal Jelinek

Montanuniversität Leoben, Franz-Josef-Strasse 18, 8700 Leoben, Austria

pascal.jelinek@unileoben.ac.at

ORCID iD: 0009-0004-5877-2908

Lukas Spiegelhofer

Montanuniversität Leoben, Franz-Josef-Strasse 18, 8700 Leoben, Austria

lukas.spiegelhofer@unileoben.ac.at

ORCID iD: 0000-0003-3552-603X