engineering journals
Redefining Knowledge

uniform distribution theory

DOI: 10.66833/UDT-2026-0001
Unif. Distrib. Theory 21 (2026), no.1, 1-9

Divergent Trajectories in the $3x+1$ Problem and a S −units Equation

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ABSTRACT. In this note we assume the existence of divergent trajectories (if any) under the Collatz map and investigate the consequences of this assumption, even though the Conjecture of Collatz seems to be true. In a trajectory we show that the Diophantine exponential equation $x+y=w2^n$ in the integers x,y,w,n plays an important role. In this context the recent strong finiteness theorem of Bennett and Billerey on S-units has consequences on the set of prime numbers generated by a collection of local minima, called wave-minima (defined ad hoc for simplicity's sake) in a divergent trajectory. It allows to define canonically a density of a trajectory and some conjectures and open problems related to these prime numbers.

Communicated by Jean-Paul Allouche

1. Introduction

The Collatz function $T: \mathbb{N} \to \mathbb{N}$ is defined by

$$T(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \bmod 2, \\ (3n+1)/2 & \text{if } n \equiv 1 \bmod 2. \end{cases} \tag{1.0.1}$$

This function became famous due to the Syracuse Conjecture, also named $3x+1$ Conjecture or Collatz Conjecture. This Conjecture has been formulated initially by Collatz as a numerical observation dating back to the 1930's [4]. Let us recall it. Denote $T^{(0)} :=$Id, $T^{(1)} := T$, $T^{(j)} := T \circ T^{(j-1)}$ for $j \geq 2$.

Conjecture 1.1. *Let $n \in \mathbb{N}$, $n \geq 1$. Then there exists $k \geq 1$ such that $T^{(k)}(n) = 1$.*

2020 Mathematics Subject Classification: 11B83, 11D61.
Keywords: Collatz Conjecture, iteration, S-units equation, prime numbers, density.

This Conjecture means that any trajectory $(T^{(j)}(n))_{j\geq 0}$, $n \geq 1$, is bounded (by an upper bound depending upon n) and ends in the trivial cycle $\{1,2\}$. We refer the reader to [1][4][8][5] for more details about the Conjecture.

The Conjecture has been recently verified on a large number of integers n by Barina [2]. Few is known on the boundedness problem of the trajectories, though Tao recently brought some light on it in [9]. Whether divergent trajectories exist or not is not known. Proving that divergent trajectories do not exist would be a formidable result.

In this note we proceed a contrario. We assume that a divergent trajectory exists and investigate the consequences of this assumption. In this divergent trajectory we show that the Diophantine exponential equation

$$x + y = w2^n \tag{1.0.2}$$

with x, y, w, n integers plays an important role in the successive iterations of the Collatz function T, in strings of successive odd integers of height greater than 2 (the height is defined below). We show that this role is due to a recent finiteness theorem obtained by Bennett and Billerey. This theorem, Theorem 3.3 in [3], is recalled below as Theorem 2.2., in which the equation (1.0.2) is an S-units equation, i.e. in which the solutions x, y, w, n have prime decompositions for which the prime factors are all in a finite set of primes S. Finiteness means that these authors, using strong theorems, prove that a subset of solutions of this equation is necessarily finite (cf Section 2 for the details).

In this note we report the rather suprising discovery that this theorem of Bennett and Billerey has consequences on the prime numbers which appear as prime factors in the prime decompositions of the iterates of T which constitute the strings of successive odd integers in the divergent trajectory. We notice that the multiplicities of these prime factors occurring in these prime decompositions do not play any role. The links between Bennett-Billerey's Theorem and the concerned prime numbers are explicited in Section 2.

As a consequence, in Section 3, a canonical definition of the density of a divergent trajectory, using these prime factors, is stated. It leads to state a Conjecture on divergent trajectories (if any) which have density zero. Some open problems are formulated in Section 3.

For doing this, two types of local minima of a given trajectory are defined. These definitions are chosen by the author ad hoc and for simplicity's sake. Let us fix the notations.

Let $n \geq 3$ be on odd integer. Let $\mathcal{T} := (T^{(j)}(n))_{j\geq 0}$ denote the trajectory of n. By definition a local minimum of $\mathcal{T}$ is either n itself or an integer $T^{(j)}(n)$ for some $j \geq 1$ such that

$$T^{(j-1)}(n) > T^{(j)}(n) < T^{(j+1)}(n).$$

A local minimum is odd. If $T^{(j)}(n)$, $j \geq 0$, is a local minimum, the iterate $T^{(j+1)}(n) = T \circ T^{(j)}(n)$ is either even or odd. By definition a *ripple* of $\mathcal{T}$ is a pair $(T^{(j)}(n), T^{(j+1)}(n))$ in which $T^{(j)}(n)$ is a local minimum and $T^{(j+1)}(n)$ is even. By definition, a *wave of height* $k \geq 2$ is a $(k+1)$-tuple of integers $(T^{(j)}(n), T^{(j+1)}(n), \ldots, T^{(j+k)}(n))$ in which $T^{(j)}(n)$ is a local minimum of $\mathcal{T}$ and for which $T^{(j+s)}(n)$, for $s = 1$ to $k-1$, are all odd with $T^{(j+k)}(n)$ even. By *wave* we mean any wave of height k, for some integer $k \geq 2$. Any local minimum μ of $\mathcal{T}$ is either the first element of a ripple or of a wave: in the first case we say that μ is a *ripple-minimum*, in the second case, that μ is a *wave-minimum*.

Let $\mathbb{P}$ denote the set of prime numbers. Let μ be a wave-minimum and denote by $(\mu = T^{(j)}(n), T^{(j+1)}(n), \ldots, T^{(j+k)}(n))$ the wave whose first element is μ. By a prime number generated by the wave-minimum μ we mean a prime number occurring either in the prime decomposition of $\mu + 1$, or in the prime decomposition of $T^{(j+s)}(n)$, for all s such that $1 \leq s \leq k$. If $\mu + 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_q^{\alpha_q}$, we denote:

$$\mathcal{Prm}(\mu) = \{p_1, p_2, \ldots, p_q\} \cup \ \cup_{s=1}^{k} \{q \in \mathbb{P} : q \text{ is a prime factor of } T^{(j+s)}(n)\} \cup \{3\}$$

the set of the prime numbers generated by μ.

Let us examplify the terminology introduced on some non-divergent trajectories.

Example 1.1. a) The trajectory of $n = 17$ under T is the following:

$$17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1 \to 2 \to 1 \to 2 \to \cdots. \quad (1.0.3)$$

It has 3 ripples, successively: $\{T^{(0)}(17) = 17, T^{(1)}(17) = 26\}$, $\{T^{(2)}(17) = 13, T^{(3)}(17) = 20\}$, $\{T^{(5)}(17) = 5, T^{(6)}(17) = 8\}$, and 3 ripple-minima greater than 1: $\{17\}, \{13\}, \{5\}$. The trajectory of 17 has no wave and no wave-minimum and infinitely many ripples $\{1,2\}$.

b) The trajectory of $n = 19$ under T contains waves and wave-minima. It is

$$19 \to 29 \to 44 \to 22 \to 11 \to 17 \to 26 \to \cdots$$

ending in the infinite sequence (1.0.3) of the iterates of 17. The two waves of the trajectory of 19, each of them of height 2, are: $\{19,29,44\}$ and $\{11,17,26\}$. The two wave-minima are $\{\mu_{1,w} = 19\}, \{\mu_{2,w} = 11\}$. Since $19 + 1 = 20 = 2^2 \times 5$ and $11 + 1 = 12 = 2^2 \times 3$, we have:

$$\mathcal{Prm}(\mu_{1,w}) = \{2,5\} \cup \{29\} \cup \{2,11\} \cup \{3\}$$

and

$$\mathcal{Prm}(\mu_{2,w}) = \{2,3\} \cup \{17\} \cup \{2,13\}.$$

Let $\mathcal{T} = (T^{(j)}(n))_{j\geq 0}$ be a divergent trajectory for some odd integer $n \geq 3$. The wave-minima of $\mathcal{T}$ uniquely define the infinite increasing sequence of integers $\mathcal{J} = \{i_j : j = 1,2,3,\ldots\})$, $0 \leq i_1 < i_2 < i_3 < \cdots$, for which the $T^{(i_j)}(n)$s are exactly the successive wave-minima of $\mathcal{T}$. We enumerate the successive wave-minima of $\mathcal{T}$ as follows:

$$\mu_{1,w} := T^{(i_1)}(n),\ \ \mu_{2,w} := T^{(i_2)}(n),\ \ \mu_{3,w} := T^{(i_3)}(n), \ldots.$$

If κ denotes a positive integer and $p_1^{\beta_1} p_2^{\beta_2} \ldots p_q^{\beta_q}$ its prime decomposition, we call β_i the p_i-adic valuation of κ, and denote it by $v_{p_i}(\kappa)$.

Theorem 1.2. *Let $\mathcal{T} = (T^{(j)}(n))_{j\geq 0}$ be a divergent trajectory for some odd integer $n \geq 3$. Then*

(i) the ripples of $\mathcal{T}$, possibly in infinite number, do not contribute to the divergence,

(ii) any wave $(\mu_{j,w} = T^{(i_j)}(n), T^{(i_j+1)}(n), \ldots, T^{(i_j+k)}(n))$, for some $j \geq 1$ and $k \geq 2$, has a finite height $k = v_2(\mu_{j,w}+1)$ which satisfies

$$T^{(v_2(\mu_{j,w}+1))}(\mu_{j,w}) = -1 + (\mu_{j,w}+1)\left(\frac{3}{2}\right)^{v_2(\mu_{j,w}+1)}, \tag{1.0.4}$$

and the number of waves of $\mathcal{T}$ is infinite,

(iii) if there exists $M \geq 2$ such that any wave $(\mu_{j,w} = T^{(i_j)}(n), T^{(i_j+1)}(n), \ldots, T^{(i_j+k)}(n))$, $j \geq 1$ and $k \geq 2$, has a height k which satisfies: $k \leq M$, the set of prime numbers generated by the wave-minima is infinite, i.e.

$$\bigcup_{j=1}^{+\infty} \mathcal{P}rm(\mu_{j,w}) \quad \text{is infinite.}$$

2. Proof of the Theorem

(i) If $T^{(j)}(n)$, for some $j \geq 0$, is a local minimum of $\mathcal{T}$ such that $(T^{(j)}(n), T^{(j+1)}(n))$ is a ripple, we have: $T^{(j+1)}(n) > T^{(j)}(n)$ and $T^{(j+2)}(n) = T^{(j+1)}(n)/2$ less than $T^{(j)}(n)$. Each time a ripple-minimum occurs, a descent of the trajectory $\mathcal{T}$ "occurs" since $T^{(j+2)}(n) < T^{(j)}(n)$.

(ii) If $\mathcal{T}$ only contains ripples and blocks of sequences of even integers successively obtained by division by 2, it could not be divergent. Then, we have at least one wave, of a certain height k which is necessarily ≥ 2, in the trajectory $\mathcal{T}$. The height k is necessarily finite by the following Proposition; it cannot be infinite.

Proposition 2.1. *Let $m \geq 3$ be an odd integer. Let $s \geq 1$ be an integer. Assume that $T^{(r)}(m)$ is odd for all $r = 0,1,\ldots,s$. Then*

$$3^r(m+1) - 2^r \;=\; T^{(r)}(m) \times 2^r \qquad \text{for } r = 0,1,2,\dots,s+1. \qquad (2.0.1)$$

Proof. If $r = 0$, we have: $(m+1) - 1 = T^{(0)}(m) = m$. Let us prove (2.0.1) by induction. Assume that (2.0.1) is true for some r such that $0 \le r \le s$. Then $T^{(r+1)}(m)$ is given by

$$T^{(r+1)}(m) = T(T^{(r)}(m)) = \frac{3T^{(r)}(m)+1}{2} = \frac{1}{2}\Big(3(\frac{3^r(m+1)-2^r}{2^r}-) + 1\Big).$$

We deduce

$$2^{r+1}T^{(r+1)}(m) = 3^{r+1}(m+1) - 3 \times 2^r + 2^r = 3^{r+1}(m+1) - 2^{r+1}.$$

And the claim.

Indeed, let us consider a wave-minimum $\mu_{j,w}$ of the trajectory $\mathcal{T}$. Since $\mu_{j,w}$ is odd, as local minimum, $\mu_{j,w}+1$ is even. Let

$$\mu_{j,w} + 1 = 2^u p_1^{\alpha_1} p_2^{\alpha_2} \dots p_q^{\alpha_q}, \qquad \text{with } u = v_2(\mu_{j,w}+1) \ge 1,$$

denote its prime decomposition ($p_i \neq p_j$ for $i \neq j$ and $p_i \neq 2$ for $1 \le i,j \le q$, α_is integers). Since $\mathcal{T}$ is assumed divergent, all the wave-minima $\mu_{j,w}$ satisfy: $\mu_{j,w} \ge 3$.

Let $(\mu_{j,w} = T^{(i_j)}(n), T^{(i_j+1)}(n), \dots, T^{(i_j+k)}(n) = T^{(k)}(\mu_{j,w}))$ be the wave of height k whose first element is the wave-minimum $\mu_{j,w}$. The integers $T^{(i_j+r)}(n)$, for $r = 0$ to $k-1$, are all odd and $T^{(i_j+k)}(n)$ is even. Let us show that $k = u$.

Indeed, Proposition 2.1. implies

$$3^r(\mu_{j,w}+1) = (T^{(r)}(\mu_{j,w}) + 1) \times 2^r, \qquad \text{for } 0 \le r \le k.$$

Hence,

$$3^r 2^{u-r} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_q^{\alpha_q} = T^{(r)}(\mu_{j,w}) + 1. \qquad (2.0.2)$$

The prime numbers occurring in the prime decomposition of any element (shifted by $+1$) of the wave $T^{(r)}(\mu_{j,w}) + 1$, $0 < r < k$, are exactly those occurring in the prime decomposition of the integer $\mu_{j,w}+1$, except possibly 3. Since $T^{(i_j+k)}(n)$ is even, $T^{(i_j+k)}(n) + 1$ is odd. We deduce $2^{u-k} = 1$ on the left-hand side of (2.0.2), i.e. $k = v_2(\mu_{1,w} + 1)$.

Therefore, from (2.0.2),

$$T^{(v_2(\mu_{j,w}+1))}(\mu_{j,w}) = \left(\frac{3}{2}\right)^{v_2(\mu_{j,w}+1)} (\mu_{j,w}+1) - 1.$$

Now, since the trajectory $\mathcal{T}$ is assumed divergent, the number of waves of $\mathcal{T}$, each of them of finite height, is necessarily infinite to have divergence. The sequence $(\mu_{j,w})_{j\geq 1}$ is infinite.

(iii) For proving (iii) we need to recall a recent theorem of Bennett and Billerey (Theorem 3.3 in [3]), with some notations, and adapt it to the $3x+1$ problem. The statement is the following.

If $S=\{p_1,p_2,\dots,p_q\}$ is a finite set of primes, we define the set of S-units to be those rational integers of the type $\pm p_1^{\alpha_1}p_2^{\alpha_2}\dots p_q^{\alpha_q}$, where the α_is are integers, and $p_i\neq p_j$ for $i\neq j$. Let a be a positive integer.

Theorem 2.2. (Bennett Billerey [3]). *There are only finitely many integer S-units x, y, w with $gcd(x,y)\leq a$ for which there exists integers $k\geq 2$ and $z\neq 0$ such that*

$$x+y=wz^k. \tag{2.0.3}$$

In the divergent trajectory $\mathcal{T}$ all the iterates of a wave-minima $\mu_{j,w}$ in the wave $(\mu_{j,w}=T^{(i_j)}(n),T^{(i_j+1)}(n),\dots,T^{(i_j+k)}(n)=T^{(k)}(\mu_{j,w}))$ are linked by the same S-units equation, which is of the type (2.0.3), that is

$$3^r(\mu_{j,w}+1)-2^r = T^{(r)}(\mu_{j,w})\times 2^r \qquad \text{for } r=0,1,2,\dots,\ k=v_2(\mu_{j,w}+1), \tag{2.0.4}$$

for which

$$x=3^r(\mu_{j,w}+1), \qquad y=-2^r,$$

$$w=T^{(r)}(\mu_{j,w}), \qquad z=2, \qquad k\geq 2.$$

We have: $\gcd(x,y)=2^r$. Here, the set S is equal to $\mathcal{P}rm(\mu_{j,w})$ and depends upon the index j of the wave-minimum $\mu_{j,w}$. The triples $(3^r(\mu_{j,w}+1),\ -2^r,\ T^{(r)}(\mu_{j,w}))$, $0\leq r\leq v_2(\mu_{j,w}+1)$, are solutions of the finite set of the S-units x,y,w with $\gcd(x,y)\leq 2^{v_2(\mu_{j,w}+1)}$ for which there exists $k\geq 2$ such that

$$x+y=w\,2^k.$$

Let us assume that any wave has a height bounded from above by an integer M. If X is a finite set of prime numbers, let us denote by U_X the set of the X-units x,y,w with $\gcd(x,y)\leq 2^M$ for which there exists $k\geq 2$ such that

$$x+y=w\,2^k.$$

By Theorem 2.2. the set U_X is finite.

To prove (iii) we assume that $\bigcup_{j=1}^{+\infty} \mathcal{P}rm(\mu_{j,w})$ is finite. Then we show that this assumption leads to a contradiction.

Let us take:

$$X := \bigcup_{j=1}^{+\infty} \mathcal{P}rm(\mu_{j,w}).$$

The set X is finite, and for any $j \geq 1$, $S := \mathcal{P}rm(\mu_{j,w})$ is such that

$$S \subset X \quad \text{implies} \quad U_S \subset U_X.$$

Therefore, all the triples $(3^r(\mu_{j,w}+1),\ -2^r,\ T^{(r)}(\mu_{j,w}))$, $j \geq 1$, $0 \leq r \leq v_2(\mu_{j,w}+1)$, belong to U_X. By (ii), it is an infinite collection of triples of integers in the finite set U_X. We deduce that there exists integers j_1 and j_2, $j_1 \neq j_2$, and $r_1, r_2 \leq M$, such that

$$(3^{r_1}(\mu_{j_1,w}+1),\ -2^{r_1},\ T^{(r_1)}(\mu_{j_1,w})) = (3^{r_2}(\mu_{j_2,w}+1),\ -2^{r_2},\ T^{(r_2)}(\mu_{j_2,w})).$$

We deduce: $r_1 = r_2$. Considering the first coordinates, we would have $\mu_{j_1,w} = \mu_{j_2,w}$. This equality means that the trajectory $\mathcal{T}$ would possess a cycle at infinity, which would be different from the trivial cycle $\{1,2\}$. The trajectory $\mathcal{T}$ would be bounded. But the trajectory $\mathcal{T}$ is assumed divergent. Contradiction. We deduce (iii).

Remark 2.3. The form of the equation (2.0.4) is common to all the waves. The only difference from one wave to another one comes from the set S of prime numbers, on which the integers x, y, w are decomposed.

3. Density and Open Problems

With the notations introduced above, let us show that the density of the divergent trajectory $\mathcal{T} = (T^{(j)}(n))_{j \geq 0}$, for $n \geq 3$ odd, (if any) can be defined using the sets of the prime numbers generated by the wave-minima of $\mathcal{T}$.

It can be done as follows: denote

$$I_j := \max \mathcal{P}rm(\mu_{j,w}), \qquad j \geq 1.$$

Then we define:

$$\delta_{\mathcal{T}} := \limsup_{X \to \infty} \frac{1}{\pi(X)} \# \left(\bigcup_{1 \leq j,\, I_j \leq X} \mathcal{P}rm(\mu_{j,w}) \right) \qquad \in [0,1]. \qquad (3.0.1)$$

This definition is extended to any finite trajectory, or to any trajectory ending in nontrivial cycles (cf [5][6]): in both cases, $\delta_{\mathcal{T}} = 0$ since the sets of prime numbers generated by the wave-minima are finite. We call $\delta_{\mathcal{T}}$ the density of $\mathcal{T}$.

As a first step, in the context of rarefied prime numbers generated by the wave-minima, a reasonable conjecture towards a partial proof of the Conjecture of Collatz would be the following.

Conjecture 3.1. Let $\mathcal{T} = (T^{(j)}(n))_{j\geq 0}$ be the trajectory of the odd integer $n \geq 3$ under the Collatz map T. Then

$$\delta_{\mathcal{T}} = 0 \quad \Rightarrow \quad \mathcal{T} \text{ is finite or ends in a nontrivial cycle.}$$

The second step, towards a proof of the Conjecture of Collatz, would be to prove that the set of trajectories $\mathcal{T}$ having a density $\delta_{\mathcal{T}} > 0$ is empty.

The combinatorics of trajectories under T in rational base $3/2$ has been shown to be simple in [7]. It is particularly the case for strings of successive odd iterates in a given trajectory. Therefore, Theorem 1.2. may justify the need to study the prime numbers in rational base $3/2$, generated by the local minima along a trajectory.

As a consequence of Theorem 1.2., considering trajectories which possess waves of arbitrarily large height, we formulate the following conjecture.

Conjecture 3.2. The set of divergent trajectories having the set of prime numbers generated by the wave-minima

$$\bigcup_{j=1}^{+\infty} \mathcal{P}rm(\mu_{j,w})$$

finite is empty.

Acknowledgements

The author thanks Gökhan Soydan for useful discussions on Diophantine exponential equations and Shalom Eliahou for his comments.

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Received January 15, 2025
Accepted July 30, 2025
Published June 1, 2026

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