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Hankel and Hermitian–Toeplitz Determinants for Einstein-Type Sakaguchi Functions

Erhan Deniz[a], Elif Ayşegül Çin[a]

[a]*Department of Mathematics, Faculty of Science and Letters, Kafkas University, Kars, Türkiye*

Abstract. In this paper, we introduce and investigate two Sakaguchi-type subclasses of normalized analytic functions which are generated by subordination to the Einstein function

$$E(z) = \frac{z}{e^z - 1}.$$

The first class is defined through the Sakaguchi starlikeness expression, whereas the second one is obtained from the corresponding Sakaguchi convexity expression; accordingly, these classes are denoted by $\mathcal{S}_E^*$ and C_E, respectively. For functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to these classes, we determine bounds for the first three non-trivial Taylor coefficients and then use these coefficient relations to estimate the second-order Hankel determinants $H_{2,1}(f)$ and $H_{2,2}(f)$. We also obtain two-sided estimates for the Hermitian–Toeplitz determinants $T_{2,1}(f)$ and $T_{3,1}(f)$. The analysis is based on the Schwarz-function representation of subordination, coefficient comparison with the Einstein expansion, and the classical Carathéodory coefficient parametrization, which enables the relevant determinant functionals to be reduced to explicitly tractable functions of two real parameters.

1. Introduction and Preliminaries

Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

denote the open unit disk, and let $\mathcal{A}$ stand for the class of all functions that are analytic in $\mathbb{D}$ and normalized by

$$f(0) = 0, \qquad f'(0) = 1.$$

Thus every $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}.$$

Coefficient problems occupy a central position in geometric function theory because they connect the analytic data encoded in the Taylor coefficients with the geometric behaviour of the associated image domains.

Corresponding author: E. D.: edeniz36@gmail.com ORCID:0000-0002-9570-8583, E.A.Ç.: ORCID:0009-0000-5647-369X
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In particular, determinant functionals constructed from the Taylor coefficients provide refined nonlinear measurements of analytic and geometric structure, and they have therefore been studied extensively for starlike, convex, close-to-convex, Ma–Minda type, and Sakaguchi-type subclasses of analytic and univalent functions.

For a normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, the Hankel determinant $H_{q,n}(f)$ is defined by (see [17])

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In the present work we shall use, in particular,

$$H_{2,1}(f) = a_3 - a_2^2, \qquad H_{2,2}(f) = a_2 a_4 - a_3^2.$$

Another coefficient functional considered here is the Hermitian–Toeplitz determinant see [6]). Its second and third forms are given by

$$T_{2,1}(f) = \begin{vmatrix} 1 & \overline{a_2} \\ a_2 & 1 \end{vmatrix} = 1 - |a_2|^2$$

and

$$T_{3,1}(f) = \begin{vmatrix} 1 & \overline{a_2} & \overline{a_3} \\ a_2 & 1 & \overline{a_2} \\ a_3 & a_2 & 1 \end{vmatrix}.$$

A direct expansion gives

$$T_{3,1}(f) = 1 - 2|a_2|^2 + 2\,\mathrm{Re}\Big(a_2^2 \overline{a_3}\Big) - |a_3|^2.$$

Sakaguchi functions [18] form a classical family related to starlikeness with respect to symmetric points. A function $f \in \mathcal{A}$ is said to be Sakaguchi starlike if

$$\mathrm{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0, \qquad z \in \mathbb{D}.$$

and the corresponding Sakaguchi convex condition is expressed through

$$\mathrm{Re}\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \qquad z \in \mathbb{D}.$$

Recently many authors have studied on Hankel determinants and Hermitian–Toeplitz determinants by using different methods (see, [1, 4, 5, 7–12, 14–16, 19]). Motivated by these symmetric-point conditions and by recent interest in coefficient determinants generated by special analytic functions, we introduce in this paper two subclasses obtained by subordinating the above Sakaguchi expressions to the Einstein function

$$E(z) = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots.$$

Although the Einstein expansion has no z^3 term, the passage through a Schwarz function produces nontrivial third-order coefficient contributions, and this feature is essential in estimating the determinants involving a_4.

Let $\mathcal{P}$ denote the Carathéodory class consisting of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

which are analytic in $\mathbb{D}$ and satisfy $\mathrm{Re}\, p(z) > 0$ in $\mathbb{D}$. The following lemmas will be used throughout the paper.

Lemma 1.1. *[2] If $p \in \mathcal{P}$, then*

$$|p_n| \leq 2, \qquad n \geq 1.$$

Lemma 1.2. *[3] If $p \in \mathcal{P}$, then, for every complex number ρ,*

$$|p_2 - \rho p_1^2| \leq 2\max\{1, |2\rho - 1|\}.$$

Lemma 1.3. *[13] If $p \in \mathcal{P}$, then there exist complex numbers δ and η, with $|\delta| \leq 1$ and $|\eta| \leq 1$, such that*

$$2p_2 = p_1^2 + (4 - p_1^2)\delta$$

and

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)\delta - p_1(4 - p_1^2)\delta^2 + 2(4 - p_1^2)(1 - |\delta|^2)\eta.$$

In the applications of Lemma 1.3, since the involved estimates are rotationally invariant, we may assume that

$$p_1 = c, \qquad 0 \leq c \leq 2,$$

and we shall write

$$|\delta| = \mu, \qquad 0 \leq \mu \leq 1.$$

2. Einstein-type Sakaguchi Classes and Coefficient Relations

Definition 2.1. *A function $f \in \mathcal{A}$ is said to belong to the Einstein-type Sakaguchi starlike class $\mathcal{S}_E^*$ if*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec E(z), \qquad z \in \mathbb{D}.$$

Equivalently, there exists a Schwarz function w, satisfying $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{D}$, such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = E(w(z)).$$

Definition 2.2. *A function $f \in \mathcal{A}$ is said to belong to the Einstein-type Sakaguchi convex class C_E if*

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec E(z), \qquad z \in \mathbb{D},$$

or, equivalently, if there exists a Schwarz function w such that

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} = E(w(z)).$$

We first record the coefficient relations that will be repeatedly used in the sequel. If w is the Schwarz function associated with the above subordination and

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

then $p \in \mathcal{P}$ and

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right)z^3 + \cdots.$$

Substitution into the Einstein expansion gives

$$E(w(z)) = 1 - \frac{p_1}{4}z + \left(\frac{7p_1^2}{48} - \frac{p_2}{4}\right)z^2 + \left(-\frac{p_1^3}{12} + \frac{7p_1 p_2}{24} - \frac{p_3}{4}\right)z^3 + \cdots. \tag{1}$$

For $f \in \mathcal{S}_E^*$, comparison of (1) with

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2z + 2a_3z^2 + (4a_4 - 2a_2a_3)z^3 + \cdots$$

yields

$$a_2 = -\frac{p_1}{8}, \qquad a_3 = \frac{7p_1^2}{96} - \frac{p_2}{8}, \qquad a_4 = -\frac{39p_1^3}{1536} + \frac{31p_1p_2}{384} - \frac{p_3}{16}. \tag{2}$$

For $f \in C_E$, comparison of (1) with

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 1 + 4a_2z + 6a_3z^2 + (16a_4 - 12a_2a_3)z^3 + \cdots$$

gives

$$a_2 = -\frac{p_1}{16}, \qquad a_3 = \frac{7p_1^2}{288} - \frac{p_2}{24}, \qquad a_4 = -\frac{39p_1^3}{6144} + \frac{31p_1p_2}{1536} - \frac{p_3}{64}. \tag{3}$$

3. Results for the Class $\mathcal{S}_E^*$

Theorem 3.1. *Let $f \in \mathcal{S}_E^*$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$. Then*

$$|a_2| \le \frac{1}{4}, \qquad |a_3| \le \frac{1}{4}, \qquad |a_4| \le \frac{1}{8}.$$

Proof. The first identity in (2) and Lemma 1.1 immediately give

$$|a_2| = \frac{|p_1|}{8} \le \frac{1}{4}.$$

Moreover,

$$a_3 = -\frac{1}{8}\left(p_2 - \frac{7}{12}p_1^2\right),$$

and Lemma 1.2, used with $\rho = 7/12$, gives

$$\left|p_2 - \frac{7}{12}p_1^2\right| \le 2,$$

so that $|a_3| \le 1/4$.

It remains to justify the estimate for a_4 without suppressing the Carathéodory parametrization. By Lemma 1.3, with $p_1 = c$, $0 \le c \le 2$, and $|\delta| = \mu$, $0 \le \mu \le 1$, the last formula in (2) becomes

$$a_4 = -\frac{c^3}{1536} + \frac{7c(4-c^2)}{768}\delta + \frac{c(4-c^2)}{64}\delta^2 - \frac{(4-c^2)(1-\mu^2)}{32}\eta.$$

Hence

$$|a_4| \le \Phi(c,\mu) := \frac{c^3}{1536} + \frac{7c(4-c^2)}{768}\mu + \frac{c(4-c^2)}{64}\mu^2 + \frac{(4-c^2)(1-\mu^2)}{32}.$$

Equivalently,

$$\Phi(c,\mu) = \frac{c^3}{1536} + \frac{4-c^2}{32} + \frac{7c(4-c^2)}{768}\mu - \frac{(4-c^2)(2-c)}{64}\mu^2.$$

For each fixed $c \in [0,2)$ this is a concave quadratic polynomial in μ, and its maximum on $[0,1]$ is therefore attained either at an endpoint or at the critical point

$$\mu_0 = \frac{7c}{24(2-c)}$$

whenever this point belongs to $[0,1]$. At the endpoints we have

$$\Phi(c,0)=\frac{c^3}{1536}+\frac{4-c^2}{32}\le\frac{1}{8}$$

and

$$\Phi(c,1)=\frac{c^3}{1536}+\frac{7c(4-c^2)}{768}+\frac{c(4-c^2)}{64}\le\frac{1}{8},$$

where the second inequality follows from

$$\frac{1}{8}-\Phi(c,1)=\frac{37c^3-152c+192}{1536}\ge 0 \qquad (0\le c\le 2).$$

If $\mu_0\in[0,1]$, then necessarily $0\le c\le 48/31$, and direct substitution gives

$$\Phi(c,\mu_0)=\frac{73c^3-1054c^2+4608}{36864}\le\frac{1}{8}.$$

Thus $|a_4|\le 1/8$, as required. □

Theorem 3.2. *If $f\in\mathcal{S}_E^*$, then*

$$|H_{2,1}(f)|\le\frac{1}{4}.$$

Proof. Using (2), we obtain

$$H_{2,1}(f)=a_3-a_2^2=-\frac{1}{8}\left(p_2-\frac{11}{24}p_1^2\right).$$

Since Lemma 1.2 with $\rho=11/24$ gives

$$\left|p_2-\frac{11}{24}p_1^2\right|\le 2,$$

it follows directly that

$$|H_{2,1}(f)|\le\frac{1}{8}\cdot 2=\frac{1}{4}.$$

□

Theorem 3.3. *If $f\in\mathcal{S}_E^*$, then*

$$|H_{2,2}(f)|\le\frac{1}{16}.$$

Proof. From (2), a straightforward calculation yields

$$H_{2,2}(f)=a_2a_4-a_3^2=-\frac{79p_1^4}{36864}+\frac{25p_1^2p_2}{3072}-\frac{p_2^2}{64}+\frac{p_1p_3}{128}.$$

Applying Lemma 1.3 with $p_1=c$ and $|\delta|=\mu$, and then using the triangle inequality, we obtain

$$|H_{2,2}(f)|\le F(c,\mu),$$

where

$$F(c,\mu)=\frac{c^4}{36864}+\frac{c^2(4-c^2)}{6144}\mu+\frac{(4-c^2)^2}{256}\mu^2+\frac{c^2(4-c^2)}{512}\mu^2+\frac{c(4-c^2)(1-\mu^2)}{256}.$$

For fixed $c\in[0,2]$,

$$\frac{\partial F}{\partial\mu}=(4-c^2)\left\{\frac{c^2}{6144}+\frac{8-c^2-2c}{256}\mu\right\}\ge 0,$$

because $8 - c^2 - 2c = (2-c)(c+4) \geq 0$ on $[0,2]$. Therefore $F(c,\mu) \leq F(c,1) = G(c)$, where

$$G(c) = \frac{67c^4}{36864} - \frac{35c^2}{1536} + \frac{1}{16}.$$

Since

$$G'(c) = \frac{67c^3}{9216} - \frac{35c}{768} < 0 \qquad (0 < c < 2),$$

the function G decreases on $[0,2]$, and hence its maximum is attained at $c = 0$. Consequently,

$$|H_{2,2}(f)| \leq G(0) = \frac{1}{16}.$$

□

Theorem 3.4. *If $f \in \mathcal{S}^*_E$, then*

$$\frac{15}{16} \leq T_{2,1}(f) \leq 1.$$

Proof. Since $T_{2,1}(f) = 1 - |a_2|^2$ and $a_2 = -p_1/8$, we have, with $p_1 = c$ and $0 \leq c \leq 2$,

$$T_{2,1}(f) = 1 - \frac{c^2}{64}.$$

This decreasing function assumes its largest value at $c = 0$ and its smallest value at $c = 2$, which gives

$$1 - \frac{4}{64} \leq T_{2,1}(f) \leq 1.$$

□

Theorem 3.5. *If $f \in \mathcal{S}^*_E$, then*

$$\frac{253}{288} \leq T_{3,1}(f) \leq 1.$$

Proof. Using (2) together with $2p_2 = p_1^2 + (4 - p_1^2)\delta$, we get

$$T_{3,1}(f) = 1 - \frac{p_1^2}{32} + \frac{p_1^4}{4608} - \frac{p_1^2(4-p_1^2)}{1536}\operatorname{Re}(\delta) - \frac{(4-p_1^2)^2}{256}|\delta|^2.$$

Set $p_1 = c$ and $|\delta| = \mu$, where $0 \leq c \leq 2$ and $0 \leq \mu \leq 1$. Since $-\operatorname{Re}(\delta) \leq \mu$, the upper estimate follows from

$$T_{3,1}(f) \leq F(c,\mu) := 1 - \frac{c^2}{32} + \frac{c^4}{4608} + \frac{c^2(4-c^2)}{1536}\mu - \frac{(4-c^2)^2}{256}\mu^2.$$

For each fixed c, the function $F(c,\mu)$ is concave in μ; its possible interior critical point is

$$\mu_0 = \frac{c^2}{12(4-c^2)}.$$

At this point,

$$F(c,\mu_0) = \frac{(c^2-64)^2}{4096} \leq 1,$$

while on the boundary one has

$$F(c,0) = 1 - \frac{c^2}{32} + \frac{c^4}{4608} \leq 1,$$

$$F(c,1) = 1 - \frac{c^2}{32} + \frac{c^4}{4608} + \frac{c^2(4-c^2)}{1536} - \frac{(4-c^2)^2}{256} \leq 1,$$

because $1 - F(c,1) = (5c^4 - 3c^2 + 72)/1152 > 0$, and also $F(0,\mu) = 1 - \mu^2/16 \leq 1$. Hence $T_{3,1}(f) \leq 1$.

For the lower estimate, using $\mathrm{Re}(\delta) \leq \mu$ and $\mu \leq 1$, we obtain

$$T_{3,1}(f) \geq G(c) := 1 - \frac{c^2}{32} + \frac{c^4}{4608} - \frac{c^2(4-c^2)}{1536} - \frac{(4-c^2)^2}{256}.$$

Since

$$G'(c) = -\frac{c}{192} - \frac{7c^3}{576} < 0 \qquad (0 < c < 2),$$

G is decreasing on $[0,2]$, and therefore

$$T_{3,1}(f) \geq G(2) = 1 - \frac{1}{8} + \frac{1}{288} = \frac{253}{288}.$$

□

4. Results for the Class C_E

Theorem 4.1. *Let $f \in C_E$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$|a_2| \leq \frac{1}{8}, \qquad |a_3| \leq \frac{1}{12}, \qquad |a_4| \leq \frac{1}{32}.$$

Proof. The first two estimates follow from (3) exactly as in the starlike case. Indeed,

$$|a_2| = \frac{|p_1|}{16} \leq \frac{1}{8}$$

and

$$a_3 = -\frac{1}{24}\left(p_2 - \frac{7}{12}p_1^2\right),$$

so that Lemma 1.2 gives $|a_3| \leq 1/12$.

For a_4, Lemma 1.3 gives

$$a_4 = -\frac{c^3}{6144} + \frac{7c(4-c^2)}{3072}\delta + \frac{c(4-c^2)}{256}\delta^2 - \frac{(4-c^2)(1-\mu^2)}{128}\eta,$$

where $p_1 = c$, $0 \leq c \leq 2$, and $|\delta| = \mu$, $0 \leq \mu \leq 1$. Thus

$$|a_4| \leq \frac{c^3}{6144} + \frac{7c(4-c^2)}{3072}\mu + \frac{c(4-c^2)}{256}\mu^2 + \frac{(4-c^2)(1-\mu^2)}{128}.$$

The right-hand side is exactly one fourth of the corresponding majorant used in the proof of Theorem 3.1; consequently it does not exceed $1/32$. This proves the asserted coefficient estimates. □

Theorem 4.2. *If $f \in C_E$, then*

$$|H_{2,1}(f)| \leq \frac{1}{12}.$$

Proof. By (3),

$$H_{2,1}(f) = a_3 - a_2^2 = -\frac{1}{24}\left(p_2 - \frac{47}{96}p_1^2\right).$$

Lemma 1.2, applied with $\rho = 47/96$, implies

$$\left|p_2 - \frac{47}{96}p_1^2\right| \leq 2,$$

and hence $|H_{2,1}(f)| \leq 1/12$. □

Theorem 4.3. *If $f \in C_E$, then*

$$|H_{2,2}(f)| \leq \frac{1}{144}.$$

Proof. Using (3), we first obtain

$$H_{2,2}(f) = -\frac{515p_1^4}{2654208} + \frac{169p_1^2p_2}{221184} - \frac{p_2^2}{576} + \frac{p_1p_3}{1024}.$$

After substituting the parametrization in Lemma 1.3, with $p_1 = c$ and $|\delta| = \mu$, the triangle inequality yields

$$|H_{2,2}(f)| \leq F(c, \mu),$$

where

$$F(c,\mu) = \frac{5c^4}{2654208} + \frac{c^2(4-c^2)}{442368}\mu + \frac{(4-c^2)^2}{2304}\mu^2 + \frac{c^2(4-c^2)}{4096}\mu^2 + \frac{c(4-c^2)(1-\mu^2)}{2048}.$$

For fixed $c \in [0, 2]$,

$$\frac{\partial F}{\partial \mu} = (4-c^2)\left\{\frac{c^2}{442368} + \frac{64-18c-7c^2}{18432}\mu\right\} \geq 0,$$

since $64 - 18c - 7c^2 \geq 0$ on $[0, 2]$. Hence $F(c, \mu) \leq F(c, 1) = G(c)$, where

$$G(c) = \frac{503c^4}{2654208} - \frac{275c^2}{110592} + \frac{1}{144}.$$

As

$$G'(c) = \frac{503c^3}{663552} - \frac{275c}{55296} < 0 \qquad (0 < c < 2),$$

the maximum of G on $[0, 2]$ is $G(0) = 1/144$. Therefore

$$|H_{2,2}(f)| \leq \frac{1}{144}.$$

□

Theorem 4.4. *If $f \in C_E$, then*

$$\frac{63}{64} \leq T_{2,1}(f) \leq 1.$$

Proof. Since $a_2 = -p_1/16$, we have

$$T_{2,1}(f) = 1 - |a_2|^2 = 1 - \frac{|p_1|^2}{256}.$$

Taking $p_1 = c$, $0 \leq c \leq 2$, gives

$$T_{2,1}(f) = 1 - \frac{c^2}{256},$$

and the desired two-sided estimate follows immediately by evaluating this decreasing function at $c = 0$ and $c = 2$. □

Theorem 4.5. *If $f \in C_E$, then*

$$\frac{20093}{20736} \leq T_{3,1}(f) \leq 1.$$

Proof. Using (3) and $2p_2 = p_1^2 + (4 - p_1^2)\delta$, we get

$$T_{3,1}(f) = 1 - \frac{p_1^2}{128} + \frac{5p_1^4}{331776} - \frac{p_1^2(4-p_1^2)}{55296}\operatorname{Re}(\delta) - \frac{(4-p_1^2)^2}{2304}|\delta|^2.$$

Let $p_1 = c$ and $|\delta| = \mu$, where $0 \le c \le 2$ and $0 \le \mu \le 1$. Since $-\operatorname{Re}(\delta) \le \mu$,

$$T_{3,1}(f) \le F(c,\mu) := 1 - \frac{c^2}{128} + \frac{5c^4}{331776} + \frac{c^2(4-c^2)}{55296}\mu - \frac{(4-c^2)^2}{2304}\mu^2.$$

The function $F(c,\mu)$ is concave in μ, and its possible interior critical point is

$$\mu_0 = \frac{c^2}{48(4-c^2)}.$$

At this point,

$$F(c,\mu_0) = \frac{(c^2-256)^2}{65536} \le 1,$$

whereas

$$F(c,0) = 1 - \frac{c^2}{128} + \frac{5c^4}{331776} \le 1$$

and

$$F(c,1) \le 1,$$

because

$$1 - F(c,1) = \frac{145c^4 + 1416c^2 + 2304}{331776} > 0.$$

Thus $T_{3,1}(f) \le 1$.

For the lower estimate, using $\operatorname{Re}(\delta) \le \mu$ and $\mu \le 1$, we derive

$$T_{3,1}(f) \ge G(c) := 1 - \frac{c^2}{128} + \frac{5c^4}{331776} - \frac{c^2(4-c^2)}{55296} - \frac{(4-c^2)^2}{2304}.$$

Since

$$G'(c) = -\frac{61c}{6912} - \frac{133c^3}{82944} < 0 \qquad (0 < c < 2),$$

G is decreasing on $[0,2]$, and hence

$$T_{3,1}(f) \ge G(2) = 1 - \frac{1}{32} + \frac{5}{20736} = \frac{20093}{20736}.$$

□

5. Concluding remarks

The preceding results show that the Einstein-type Sakaguchi starlike and convex classes admit explicit coefficient and determinant estimates that can be obtained through a unified Carathéodory-parametric approach. In comparison with the starlike class $\mathcal{S}_E^*$, the convex class $\mathcal{C}_E$ produces systematically narrower bounds for the initial coefficients and for the Hankel-type quantities considered here, which is consistent with the additional rigidity imposed by the convexity-type differential expression. The method is flexible enough to be adapted to other subordinating functions whose Taylor coefficients are explicitly known, and it may therefore be useful for further investigations of Hankel, Toeplitz, and related nonlinear coefficient functionals in symmetric-point subclasses of analytic functions.

Conflict of interest

The authors declare that they have no conflict of interest.

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