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Refined Fractional Milne-Type Inequalities via Strongly $s-$Convexity with Enhanced Error Bounds

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Abstract. This study introduces a novel family of refined Milne-type inequalities formulated via Riemann–Liouville fractional integral operators. Utilizing an appropriate integral identity for twice continuously differentiable functions, improved estimates are established for mappings whose second derivatives fulfill a strong s-convexity condition. Unlike existing contributions, the obtained results provide improved error estimates by incorporating the modulus parameter and exploiting the structure of strongly s-convexity more effectively. Several inequalities are established via Hölder and Young integral inequalities, leading to tighter upper bounds compared to previously known results. Furthermore, special cases are discussed to demonstrate the consistency of the results with classical inequalities, including Simpson- and Newton-type estimates. The findings presented here extend and complement earlier studies in fractional inequality theory and contribute to a more precise understanding of numerical integration errors under generalized convexity assumptions.

1. Introduction

Numerical integration constitutes a fundamental component of scientific computation, particularly in situations where closed-form evaluation of definite integrals is unattainable due to the intricate nature of the integrand. In such contexts, approximation schemes offer effective and computationally feasible alternatives. Accordingly, a substantial body of research has been devoted to enhancing the precision of these methods while deriving rigorous and reliable error bounds. A key strategy in this direction involves the application of analytical inequalities tailored to specific function classes—such as convex, bounded, and Lipschitz-continuous functions—which facilitate the development of sharper and more informative error estimates.

In this work, numerical integration techniques are investigated from the perspective of these structural function properties. Special emphasis is placed on functions whose derivatives satisfy various convexity-type conditions, enabling a detailed examination of how such characteristics influence approximation errors. By establishing a clearer connection between functional behavior and error estimates, this study aims to reinforce the theoretical foundations of numerical integration while simultaneously improving its practical performance. Consequently, the results contribute both to the refinement of existing approximation

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methods and to a more comprehensive understanding of their accuracy and efficiency.
The formulations of Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule are given as follows[11].

$$\int_{\hbar_1}^{\hbar_2} \chi(\mu)d\mu \approx \frac{\hbar_2-\hbar_1}{6}[\chi(\hbar_1)+4\chi(\frac{\hbar_1+\hbar_2}{2})+\chi(\hbar_2)] \tag{1}$$

$$\int_{\hbar_1}^{\hbar_2} \chi(\mu)d\mu \approx \frac{\hbar_2-\hbar_1}{8}[\chi(\hbar_1)+3\chi(\frac{2\hbar_1+\hbar_2}{3})+3\chi(\frac{\hbar_1+2\hbar_2}{3})+\chi(\hbar_2)] \tag{2}$$

Relations (1) and (2) are valid for every function χ whose fourth derivative exists and is continuous on the interval $[\hbar_1, \hbar_2]$:

The classical expression of Simpson's inequality can be stated as follows.

Theorem 1.1. *Assume that* $\chi : [\hbar_1,\hbar_2] \to \mathbb{R}$ *is a function whose fourth derivative exists and is continuous on the interval* $(\hbar_1, \hbar_2)$, $\left\|\chi^{(4)}\right\|_\infty = \sup_{\mu\in(\hbar_1,\hbar_2)} |\chi^{(4)}(\mu)| < \infty$, *the following inequality is satisfied:*

$$\left|\frac{1}{6}\left[\chi(\hbar_1)+4\chi(\frac{\hbar_1+\hbar_2}{2})+\chi(\hbar_2)\right]-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\chi(\mu)d\mu\right| \le \frac{1}{2880}\left\|\chi^{(4)}\right\|_\infty (\hbar_2-\hbar_1)^4.$$

A Simpson-type inequality was originally established by Sarıkaya and co-authors through the use of convexity assumptions, thereby providing a foundational result within this framework [12]. Within the setting of Riemann–Liouville fractional integral operators, Simpson-type inequalities admit a natural classification into three principal forms, distinguished by the structure of their fractional integral representations and the manner in which the kernel functions are incorporated. Each class reflects a different analytical approach to approximating integrals and estimating the associated error terms under fractional frameworks. These formulations not only extend the classical Simpson inequality but also provide greater flexibility in capturing nonlocal behaviors inherent in fractional calculus. The development, refinement, and comparative analysis of these distinct forms have been thoroughly investigated in the contributions documented in [13]–[15], where various generalizations and sharper bounds are established. These studies have significantly advanced the generalization of Simpson's inequality within the framework of fractional analysis, providing a broader theoretical foundation for its application. In particular, they demonstrate that such inequalities can be effectively adapted to a variety of fractional integral operators, thereby highlighting their versatility and relevance in different fractional settings. Moreover, the studies reported in [16]–[18] place particular emphasis on the development and analysis of Simpson-type inequalities tailored to twice differentiable functions, yielding refined estimates that exploit the regularity of second-order derivatives. These contributions provide a thorough investigation of Simpson's inequality in the setting of twice differentiable functions, with particular attention to deriving and analyzing specialized forms adapted to this class. By exploiting second-order differentiability, they establish more refined variants of the inequality and examine their corresponding bounds in detail. As a result, the applicability of Simpson-type inequalities is substantially broadened, enabling the derivation of sharper and more function-specific estimates for this category of mappings.

The classical Newton inequality occupies a central role in mathematical analysis and can be formulated as follows.

Theorem 1.2. *[11] If* $\chi : [\hbar_1,\hbar_2] \to \mathbb{R}$ *represents a function with a continuous fourth derivative defined over* $(\hbar_1, \hbar_2)$, *and* $\left\|\chi^{(4)}\right\|_\infty = \sup_{\mu\in(\hbar_1,\hbar_2)} |\chi^{(4)}(\mu)| < \infty$, *then the inequality presented below is valid:*

$$\begin{aligned}&\left|\frac{1}{8}\left[\chi(\hbar_1)+3\chi(\frac{2\hbar_1+\hbar_2}{3})+3\chi(\frac{\hbar_1+2\hbar_2}{3})+\chi(\hbar_2)\right]-\frac{1}{\hbar_2-\hbar_1}\int_{\hbar_1}^{\hbar_2}\chi(\mu)d\mu\right|\\ \le\ & \frac{1}{6480}\left\|\chi^{(4)}\right\|_\infty (\hbar_2-\hbar_1)^4.\end{aligned}$$

The studies reported in [19]-[21] formulate Newton-type inequalities in the setting of local fractional integrals by systematically incorporating convexity assumptions. Through this approach, the classical scope of Newton's inequality is extended to fractional analytical frameworks, yielding a broader and more flexible structure for inequality analysis. These contributions not only generalize the traditional form of the inequality but also introduce new perspectives that enhance its applicability and theoretical significance within fractional calculus.In [22], Newton-type inequalities were rigorously derived for Riemann–Liouville fractional integrals, providing one of the earliest systematic treatments of this problem. The results obtained therein establish a solid theoretical foundation for the study of such inequalities in fractional settings and serve as a cornerstone for subsequent developments in the literature.This contribution represents a notable advancement in fractional calculus and has become a key reference point for subsequent research in the area. Building on these foundational results, later studies have primarily concentrated on the formulation of Newton-type inequalities within the framework of Riemann–Liouville fractional integrals, along with a systematic analysis of their validity under various assumptions and functional settings [23, 24].These studies have made substantial contributions to the advancement of fractional integral theory by strengthening its theoretical framework and addressing previously unresolved aspects in the literature. In particular, they have helped to bridge existing gaps and have provided a more comprehensive understanding of the underlying structures and applications within this field.

In their seminal study [25], Djenaoui and Meftah developed Milne-type inequalities within a convexity-based framework, thereby establishing a noteworthy advancement in the integration of convex analysis into the theory of such inequalities. Their contributions laid the groundwork for reinterpreting classical Milne-type inequalities within a broader and more generalized analytical context. Subsequently, Budak et al. [26] extended these inequalities by embedding them within the Riemann–Liouville fractional integral framework, thereby broadening their analytical scope and enhancing their applicability in fractional settings. This extension not only enriched the theoretical basis of Milne-type inequalities but also opened new avenues for their application within the domain of fractional calculus.

Further advances were reported in [27, 28], where new fractional extensions of Milne-type inequalities were established and systematically analyzed in the context of separable convex functions, thereby enriching both their structural formulation and range of applicability. These contributions have substantially enhanced the applicability of Milne-type inequalities by providing a rigorous treatment across diverse function classes, including bounded, Lipschitz continuous, and functions of bounded variation, thereby extending their analytical scope and practical significance. For those seeking a more in-depth exploration of Milne-type inequalities, including their numerous generalizations and theoretical implications, references [29]–[31] offer comprehensive insights and valuable context.

This paper aims to derive new fractional Milne-type inequalities for functions whose second derivatives satisfy appropriate convexity conditions. To achieve this, we first establish a rigorous analytical framework by recalling the fundamental concepts of Riemann–Liouville fractional integrals, classical convexity, and second-sense s-convexity, which serve as the structural basis of our approach. Within this setting, we develop a novel class of fractional Milne-type inequalities that explicitly incorporate these convexity assumptions. The results obtained not only extend existing inequalities in the literature but also provide refined estimates and a more comprehensive perspective on the interaction between convexity properties and fractional integral operators, thereby contributing to a deeper understanding of their significance in fractional analysis.

Definition 1.3. *[8] Let $\chi \in L_1[\mu_1, \mu_2]$. For $0 < \mu_1$, the left- and right-sided Riemann-Liouville fractional integrals of order α, denoted by $J^{\alpha}_{\mu_1^+}\chi$ and $J^{\alpha}_{\mu_2^-}\chi$, respectively, are defined as follows.*

$$J^{\alpha}_{\mu_1^+}\chi(u_1) = \frac{1}{\Gamma(\alpha)} \int_{\mu_1}^{u_1} (u_1 - \zeta)^{\alpha-1}\chi(\zeta)d\zeta, \quad u_1 > \mu_1$$

and

$$J^{\alpha}_{\mu_2^-}\chi(u_1) = \frac{1}{\Gamma(\alpha)} \int_{u_1}^{\mu_2} (\zeta - u_1)^{\alpha-1}\chi(\zeta)d\zeta, \quad u_1 < \mu_2$$

where $\Gamma(\alpha) = \int_0^\infty e^{-\zeta} u^{\alpha-1} du$, $J^0_{\mu_1^+}\chi(u_1) = J^0_{\mu_2^-}\chi(u_1) = \chi(u_1)$.

It is worth noting that, in the above definition, choosing the parameter $\alpha = 1$ reduces the fractional operator to the standard (classical) integral. Hence, the given formulation can be viewed as a natural extension of the classical integral concept, ensuring consistency with the traditional case while broadening its scope to non-integer orders.

A comprehensive account of the fundamental properties and analytical characteristics of the fractional integral operator is available in the literature; for detailed discussions, we refer the reader to [2]–[8]. These sources provide rigorous treatments of its structural properties, continuity behavior, and its role within the broader framework of fractional calculus.

We proceed by introducing the notions of strongly convex and strongly $s-$convex functions, which play a central role in the subsequent analysis. These concepts extend the classical definition of convexity by incorporating additional structural conditions, thereby providing a more flexible and robust framework for the development of our results.

Definition 1.4. *[9] Let* $I \subset \mathbb{R}$ *be an interval. A function* $\chi : I \to \mathbb{R}$ *is called convex if, for every* $x, y \in I$ *and any* $\nu \in [0,1]$*, it satisfies the inequality*

$$\chi\left(\nu x + (1-\nu) y\right) \leq \nu \chi(x) + (1-\nu)\chi(y)$$

This condition expresses that the function value at any convex combination of two points does not exceed the corresponding convex combination of the function values at those points.

In [10], Boris T. Polyak introduces and formally characterizes the notions of strongly convex functions and strongly s-convex functions, which can be described as follows.

Definition 1.5. *[10] Let* $\mathbb{A} \subset \mathbb{R}$ *be an interval and let* $\chi : \mathbb{A} \to \mathbb{R}$ *be a real-valued function. The function* χ *is said to be strongly convex with modulus* $\hbar > 0$ *if, for every* $x, y \in \mathbb{A}$ *and for all* $\nu \in [0,1]$*, the inequality*

$$\chi\Big(\nu x + (1-\nu)y\Big) \leq \nu\chi(x) + (1-\nu)\chi(y) - \hbar\nu(1-\nu)(x-y)^2$$

holds.

Definition 1.6. *[10] Let* $\mathbb{A} \subset \mathbb{R}$ *be an interval and let* $\chi : \mathbb{A} \to \mathbb{R}$ *be a real-valued function. The function* χ *is said to be strongly* $s-$*convex with modulus* $\hbar > 0$ *if, for every* $x, y \in \mathbb{A}$ *and for all* $\nu \in [0,1]$*, the inequality*

$$\chi\Big(\nu x + (1-\nu)y\Big) \leq \nu^s\chi(x) + (1-\nu)^s\chi(y) - \hbar\nu(1-\nu)(x-y)^2$$

holds.

Definition 1.7. *The Gamma function is given by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-x} dx, \qquad Re(x) > 0.$$

The Beta function is given by $\beta(a,b)$ *is defined by*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt, \qquad Re(a) > 0, Re(b) > 0$$

and the incomplete beta function is given by

$$\mathcal{B}(a,b,r) = \int_0^r t^{a-1}(1-t)^{b-1} dt,$$

$Re(a) > 0, Re(b) > 0$ *and* $0 < r < 1$.

Lemma 1.8. *[1] If $\chi : [\hbar_1, \hbar_2] \longrightarrow R$ is absolutely continuous over $(\hbar_1, \hbar_2)$ and $\chi'' \in L_1([\hbar_1, \hbar_2])$, then the following holds:*

$$\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]=\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\sum_{k=1}^{4}I_k, \tag{3}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}(\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu)\chi''(\mu\hbar_2+(1-\mu)\hbar_1)d\mu,\\
I_2 &= \int_0^{\frac{1}{2}}(\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu)\chi''(\mu\hbar_1+(1-\mu)\hbar_2)d\mu,\\
I_3 &= \int_{\frac{1}{2}}^{1}(\mu^{\zeta+1}-\mu)\chi''(\mu\hbar_2+(1-\mu)\hbar_1)d\mu,\\
I_4 &= \int_{\frac{1}{2}}^{1}(\mu^{\zeta+1}-\mu)\chi''(\mu\hbar_1+(1-\mu)\hbar_2)d\mu.
\end{aligned}$$

2. Main Results

Theorem 2.1. *Assume that the hypotheses of Lemma 1.8 are fulfilled. In addition, suppose that the function $|\chi''|$ is strongly s-convex on the interval $[\hbar_1, \hbar_2]$. Then, the following result holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ &\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\times\left[\left(\left|\chi''(\hbar_1)\right|+\left|\chi''(\hbar_2)\right|\right)\times\left(\frac{(\zeta+4)(1-2^{-s-1})+3(2^{-s-1}+s+1)}{3(\zeta+1)(\zeta+2)}-\frac{1}{\zeta+s+2}\right.\right.\\
&\left.\left.B(\zeta+2,s+1)\right)-\hbar(\hbar_2-\hbar_1)^2\left[\frac{(5\zeta+53)(\zeta+3)(\zeta+4)-576}{288(\zeta+3)(\zeta+4)}\right]\right].
\end{aligned}$$

Proof. Using Lemma 1.8 and absolute value, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ &\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu\right.\\
&+\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu\\
&+\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu\\
&\left.+\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu\right].
\end{aligned}$$

Leveraging the strongly $s-$convex function property of $\left|\chi''\right|$, we derive

$$\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|$$

$$\begin{aligned}
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)\left[\mu^s\left|\chi''(\hbar_2)\right|+(1-\mu)^s\left|\chi''(\hbar_1)\right|-\hbar\mu(1-\mu)(\hbar_2-\hbar_1)^2\right]d\mu\right.\\
&+\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)\left[\mu^s\left|\chi''(\hbar_1)\right|+(1-\mu)^s\left|\chi''(\hbar_2)\right|-\hbar\mu(1-\mu)(\hbar_2-\hbar_1)^2\right]d\mu\\
&+\int_{\frac{1}{2}}^{1}\left(\mu-\mu^{\zeta+1}\right)\left[\mu^s\left|\chi''(\hbar_2)\right|+(1-\mu)^s\left|\chi''(\hbar_1)\right|-\hbar\mu(1-\mu)(\hbar_2-\hbar_1)^2\right]d\mu\\
&\left.+\int_{\frac{1}{2}}^{1}\left(\mu-\mu^{\zeta+1}\right)\left[\mu^s\left|\chi''(\hbar_1)\right|+(1-\mu)^s\left|\chi''(\hbar_2)\right|-\hbar\mu(1-\mu)(\hbar_2-\hbar_1)^2\right]d\mu\right].\\
&=\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\times\left[\left(\left|\chi''(\hbar_1)\right|+\left|\chi''(\hbar_2)\right|\right)\left(\frac{(\zeta+4)(1-2^{-s-1})+3(2^{-s-1}+s+1)}{3(\zeta+1)(\zeta+2)}-\frac{1}{\zeta+s+2}\right.\right.\\
&\left.\left.-B(\zeta+2,s+1)\right)-\hbar(\hbar_1-\hbar_2)^2\left[\frac{(5\zeta+53)(\zeta+3)(\zeta+4)-576}{288(\zeta+3)(\zeta+4)}\right]\right].
\end{aligned}$$

Remark 2.2. *[32] By specializing Theorem 2.1 to the case $s=1$, we derive the following inequality.*

$$\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|$$

$$\leq\ \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\times\left[\left(\frac{\zeta^2+15\zeta+2}{24(\zeta+2)}\right)\left[\left|\chi''(\hbar_1)\right|+\left|\chi''(\hbar_2)\right|\right]-\hbar(\hbar_1-\hbar_2)^2\left[\frac{(5\zeta+53)(\zeta+3)(\zeta+4)-576}{288(\zeta+3)(\zeta+4)}\right]\right]$$

Remark 2.3. *[32] By specializing Theorem 2.1 to the case $\zeta=s=1$, we derive the following inequality.*

$$\left|\frac{1}{(\hbar_2-\hbar_1)}\int_{\hbar_1}^{\hbar_2}\chi(\mu)d\mu-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|$$

$$\leq\frac{(\hbar_2-\hbar_1)^2}{4}\left[\frac{\left|\chi''(\hbar_1)\right|+\left|\chi''(\hbar_2)\right|}{4}-\frac{73\hbar(\hbar_2-\hbar_1)^2}{720}\right]$$

□

Theorem 2.4. *Assume that the hypotheses of Lemma 1.8 are fulfilled. In addition, suppose that the function $\left|\chi''\right|^q$ is strongly $s-$convex on the interval $[\hbar_1,\hbar_2]$. Then, the following result holds:*

$$\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|$$

$$\begin{aligned}
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{12.2^{-s-1}\left|\chi''(\hbar_2)\right|^q+12(1-2^{-s-1})\left|\chi''(\hbar_1)\right|^q-(s+1)\hbar(\hbar_1-\hbar_2)^2}{(s+1)12}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{12.2^{-s-1}\left|\chi''(\hbar_1)\right|^q+12(1-2^{-s-1})\left|\chi''(\hbar_2)\right|^q-(s+1)\hbar(\hbar_1-\hbar_2)^2}{(s+1)12}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, q>1$.

Proof. Using Lemma 1.8 and absolute value, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right| \\
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu\right. \\
&+\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu \\
&+\int_{\frac{1}{2}}^1\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu \\
&\left.+\int_{\frac{1}{2}}^1\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu\right].
\end{aligned}$$

Utilizing Hölder's inequality in the preceding inequality, we derive the following result.

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right| \\
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|^q d\mu\right)^{\frac{1}{q}}\right. \\
&+\left(\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|^q d\mu\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^1\left|\mu^{\zeta+1}-\mu\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|^q d\mu\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\mu^{\zeta+1}-\mu\right|^p d\mu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|^q d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Utilizing the strongly s−convex function $\left|\chi''\right|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ &\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\times\\
&\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\mu^s\left|\chi''(\hbar_2)\right|^q+(1-\mu)^s\left|\chi''(\hbar_1)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\mu^s\left|\chi''(\hbar_1)\right|^q+(1-\mu)^s\left|\chi''(\hbar_2)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left(\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\mu^s\left|\chi''(\hbar_2)\right|^q+(1-\mu)^s\left|\chi''(\hbar_1)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)^{\frac{1}{q}}\\
&\left.+\left(\int_{\frac{1}{2}}^1\left(\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\mu^s\left|\chi''(\hbar_1)\right|^q+(1-\mu)^s\left|\chi''(\hbar_2)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)^{\frac{1}{q}}\right]\\
=\ &\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{12.2^{-s-1}\left|\chi''(\hbar_2)\right|^q+12(1-2^{-s-1})\left|\chi''(\hbar_1)\right|^q-(s+1)\hbar(\hbar_1-\hbar_2)^2}{(s+1)12}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{12.2^{-s-1}\left|\chi''(\hbar_1)\right|^q+12(1-2^{-s-1})\left|\chi''(\hbar_2)\right|^q-(s+1)\hbar(\hbar_1-\hbar_2)^2}{(s+1)12}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

□

Remark 2.5. *[32] By specializing Theorem 2.4 to the case $s=1$, we derive the following inequality.*

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ &\frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)^{\frac{1}{p}}+\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{3\left|\chi''(\hbar_2)\right|^q+9\left|\chi''(\hbar_1)\right|^q-2\hbar(\hbar_2-\hbar_1)^2}{24}\right)^{\frac{1}{q}}+\left(\frac{3\left|\chi''(\hbar_1)\right|^q+9\left|\chi''(\hbar_2)\right|^q-2\hbar(\hbar_2-\hbar_1)^2}{24}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

Remark 2.6. *[32] By specializing Theorem 2.4 to the case $\zeta=s=1$, we derive the following inequality.*

$$
\begin{aligned}
&\left|\frac{1}{(\hbar_2-\hbar_1)}\int_{\hbar_1}^{\hbar_2}\chi(\mu)d\mu-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ &\frac{(\hbar_2-\hbar_1)^2}{4}\left[\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\mu-\mu^2\right)^p d\mu\right)^{\frac{1}{p}}+\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)^{\frac{1}{p}}\right]\\
&\times\left[\left(\frac{3\left|\chi''(\hbar_2)\right|^q+9\left|\chi''(\hbar_1)\right|^q-2\hbar(\hbar_2-\hbar_1)^2}{24}\right)^{\frac{1}{q}}+\left(\frac{3\left|\chi''(\hbar_1)\right|^q+9\left|\chi''(\hbar_2)\right|^q-2\hbar(\hbar_2-\hbar_1)^2}{24}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

Theorem 2.7. *Assume that the hypotheses of Lemma 1.8 are fulfilled. In addition, suppose that the function $\left|\chi''\right|^q$ is strongly $s-$convex on the interval $[\hbar_1, \hbar_2]$. Then, the following result holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\times\left[\frac{2}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{2}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)\right.\\
&\left.+\frac{6\left|\chi''(\hbar_2)\right|^q+6\left|\chi''(\hbar_1)\right|^q-2(s+1)\hbar(\hbar_1-\hbar_2)^2}{3(s+1)q}\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1, q>1, \hbar>0$.

Proof. Using Lemma 1.8 and absolute value, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu\right.\\
&+\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu\\
&+\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|d\mu\\
&\left.+\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|d\mu\right].
\end{aligned}$$

Utilizing Young's inequality in the preceding inequality, we derive the following result.

$$\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right|\\
\leq\ & \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)}\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|^p d\mu\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|^q d\mu\right)\right.\\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left|\mu^{\zeta+1}-\frac{\zeta+4}{3}\mu\right|^p d\mu\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|^q d\mu\right)\\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|^p d\mu\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^{1}\left|\chi''(\mu\hbar_2+(1-\mu)\hbar_1)\right|^q d\mu\right)\\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^{1}\left|\mu^{\zeta+1}-\mu\right|^p d\mu\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^{1}\left|\chi''(\mu\hbar_1+(1-\mu)\hbar_2)\right|^q d\mu\right)\right].
\end{aligned}$$

Utilizing the strongly $s-$convex function $\left|\chi''\right|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right| \\
\leq\;& \frac{(\hbar_2-\hbar_1)^2}{2(\zeta+1)} \\
&\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\mu^s\left|\chi''(\hbar_2)\right|^q+(1-\mu)^s\left|\chi''(\hbar_1)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)\right. \\
&+\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\mu^s\left|\chi''(\hbar_1)\right|^q+(1-\mu)^s\left|\chi''(\hbar_2)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right) \\
&+\frac{1}{p}\left(\int_{\frac{1}{2}}^{1}\left(\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^{1}\mu^s\left|\chi''(\hbar_2)\right|^q+(1-\mu)^s\left|\chi''(\hbar_1)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right) \\
&\left.+\frac{1}{p}\left(\int_{\frac{1}{2}}^{1}\left(\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^{1}\mu^s\left|\chi''(\hbar_1)\right|^q+(1-\mu)^s\left|\chi''(\hbar_2)\right|^q-\hbar\mu(1-\mu)(\hbar_1-\hbar_2)^2d\mu\right)\right] \\
&=\frac{(\hbar_2-\hbar_1)^2}{(\zeta+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)\right. \\
&\left.+\frac{3\left|\chi''(\hbar_2)\right|^q+3\left|\chi''(\hbar_1)\right|^q-(s+1)\hbar(\hbar_1-\hbar_2)^2}{3(s+1)q}\right]
\end{aligned}
$$

□

Remark 2.8. *[32] By specializing Theorem 2.7 to the case $s=1$, we derive the following inequality.*

$$
\begin{aligned}
&\left|\frac{\Gamma(\zeta+1)}{2(\hbar_2-\hbar_1)^{\zeta}}[J^{\zeta}_{\hbar_1^+}\chi(\hbar_2)+J^{\zeta}_{\hbar_2^-}\chi(\hbar_1)]-\frac{1}{3}[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)]\right| \\
\leq\;&\frac{(\hbar_2-\hbar_1)^2}{(\zeta+1)}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{\zeta+4}{3}\mu-\mu^{\zeta+1}\right)^p d\mu\right)+\frac{1}{q}\left(\frac{1}{\zeta}\mathcal{B}\left(p+1,\frac{p+1}{\zeta},1-\left(\frac{1}{2}\right)^{\zeta}\right)\right)\right. \\
&\left.+\frac{3\left|\chi''(\hbar_2)\right|^q+3\left|\chi''(\hbar_1)\right|^q-2\hbar(\hbar_1-\hbar_2)^2}{6q}\right]
\end{aligned}
$$

Remark 2.9. *[32] By specializing Theorem 2.4 to the case $\zeta=s=1$, we derive the following inequality.*

$$
\begin{aligned}
&\left|\frac{1}{(\hbar_2-\hbar_1)}\int_{\hbar_1}^{\hbar_2}\chi(\mu)d\mu-\frac{1}{3}\left[2\chi(\hbar_1)-\chi(\frac{\hbar_1+\hbar_2}{2})+2\chi(\hbar_2)\right]\right| \\
\leq\;&\frac{(\hbar_2-\hbar_1)^2}{2}\times\left[\frac{1}{p}\left(\int_0^{\frac{1}{2}}\left(\frac{5}{3}\mu-\mu^2\right)^p d\mu\right)+\frac{1}{q}\left(\mathcal{B}\left(p+1,p+1,\frac{1}{2}\right)\right)\right. \\
&\left.+\frac{3\left|\chi''(\hbar_2)\right|^q+3\left|\chi''(\hbar_1)\right|^q-2\hbar(\hbar_1-\hbar_2)^2}{6q}\right]
\end{aligned}
$$

3. Conclusion

This paper develops a new family of fractional Milne-type inequalities for twice continuously differentiable functions whose second derivatives fulfill strong s-convexity assumptions. The analysis is built upon a suitable integral identity, which is systematically integrated with the structure of Riemann–Liouville fractional integral operators. This synthesis yields a coherent and unified framework that facilitates the derivation of refined inequality bounds. In addition, the approach not only generalizes several existing results in the literature but also provides sharper estimates under more flexible convexity conditions, thereby enhancing the applicability of Milne-type inequalities within the setting of fractional calculus.

The novelty of the paper lies in the systematic incorporation of strong s-convexity into the analysis, which allows the derivation of more informative and structured upper bounds. In particular, the obtained inequalities explicitly reflect the influence of the convexity parameter and the modulus, providing a finer characterization of the associated error terms. The application of Hölder and Young inequalities further enhances the flexibility of the approach and yields a variety of complementary estimates.

Moreover, the presented results include several special cases that reduce to known inequalities in the literature, confirming the consistency and validity of the theoretical framework. At the same time, the general forms obtained in this study extend and refine earlier contributions related to fractional Milne-type inequalities.

In conclusion, this work contributes to the theory of fractional integral inequalities by offering a coherent methodology for handling strongly s-convex functions. The results may serve as a useful reference for further studies on generalized convexity and fractional operators, as well as for investigations related to error estimation in numerical integration.

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